Extending Two Dimensions to Three Dimensions

Unit Overview
In this unit you will study many geometric concepts related to two-dimensional and three-dimensional figures. You will derive area and perimeter formulas of polygons and circumference and area formulas of circles. You will investigate angles of a polygon and arcs and sectors of a circle. You will describe properties of prisms, pyramids, cones, cylinders, and spheres and develop and apply formulas for surface area and volume.

Key Terms
As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Academic Vocabulary
- replicate
- cross section

Math Terms
- composite figure
- density
- polygon
- interior angle
- discrete domain
- regular polygon
- equilateral
- equiangular
- exterior angle
- convex polygon
- apothem
- circumference
- sector
- concentric circles
- radian measure
- net
- face
- edge
- vertex
- oblique prism
- right prism
- oblique pyramid
- polyhedron
- vertices of the polyhedron
- cylinder
- cone
- height of cone
- sphere
- great circle
- solid of rotation
- lateral area
- total surface area
- lateral surface
- Cavalieri’s Principle
- slant height
- hemisphere
- antipodal points
- lune

ESSENTIAL QUESTIONS
- How do two-dimensional figures help you visualize three-dimensional figures?
- Why are geometric formulas useful in solving real-world problems?

EMBEDDED ASSESSMENTS
This unit has three embedded assessments, following Activities 32, 35, and 37. The assessments will give you the opportunity to apply what you have learned.

Embedded Assessment 1: Area and Perimeter p. 477
Embedded Assessment 2: Surface Area and Volume p. 523
Embedded Assessment 3: Changing Dimensions of Spheres p. 549
Write your answers on notebook paper. Show your work.

1. Solve the following equation for \( s \).
   \[
P = \frac{1}{3}r(q + s)
   \]

2. Identify the following two- and three-dimensional figures.
   a. 
   b. 
   c. 
   d. 
   e. 
   f. 

Items 3 and 4 refer to the figures below.

3. Find the measure of sides \( a \) and \( x \).
4. Name the pairs of angles that are congruent.
5. Give the area of a circle that has a circumference of \( 15\pi \) units.

6. In the figure below, a circle is inscribed in a square that has side length 8 units. Express the shaded area of the figure in terms of \( \pi \).

The figure below is composed of three geometric shapes. Use the figure for Items 7 and 8.

7. Name each of the figures and tell the formula you would use to find its area.
8. Find the area of the composite figure.
Learning Targets:

- Solve problems using the areas of rectangles, parallelograms, and composite figures.
- Use coordinates to compute perimeters and areas of figures.

SUGGESTED LEARNING STRATEGIES: Use Manipulatives, Quickwrite, RAFT

Lisa works in the billing department of A Cut Above, a company that builds custom countertops and tabletops. A new customer has contracted A Cut Above to build the tabletops for a theme restaurant, Shape Up.

Each tabletop costs $8.50 per square foot. Lisa’s job is to calculate the area, compute the charge, and bill each customer properly. The tabletops, made with laminated wood, are delivered to A Cut Above in different rectangular sizes. Before there are any cuts, a cardboard template is made to use as a guide. Lisa uses the templates to investigate the areas. Shown is one of the templates for which Lisa must find the area.

1. **a.** Find the area of this shape. Include units of measure in your answer. Describe the method used.

   **b.** How much should Lisa charge the customer for this tabletop?

In keeping with the theme of the restaurant, the customer wants to include tables in the shapes of triangles, parallelograms, and trapezoids. Lisa decides to investigate area formulas by first finding the formula for the area of a parallelogram.

2. **Use appropriate tools strategically.** Use the rectangle provided by your teacher.
   - Pick a point, not a vertex, on one side of the rectangle. Draw a segment from the point to a vertex on the opposite side of the rectangle.
   - Cut along the segment.
   - Put the two figures together to form a parallelogram.

   **a.** Explain why the area of a parallelogram is the same as the area of rectangle.

   **b.** Critique the reasoning of others. Nelson and Rashid, two tabletop designers, are discussing how to calculate the area of any parallelogram. Nelson suggests that they multiply the lengths of two consecutive sides to find the area, as they do for rectangles. Rashid claims this will not work. Explain which designer is correct and why.
Lesson 30-1
Areas of Rectangles and Parallelograms

3. A scale drawing of the parallelogram tabletop template is shown below. Determine the charge for making this tabletop. Show the calculations that led to your answer.

Lisa finds the template method somewhat time-consuming and asks Rashid if there is a different way to determine the measures of the tabletops. Rashid tells Lisa that she can use a coordinate plane to determine the perimeter and area of a tabletop.

4. A diagram showing a rectangular tabletop is plotted on the coordinate plane.

Explain how to use the coordinates to find the length and width of the rectangular tabletop. Then use the dimensions to compute the perimeter and area of the tabletop. Show the formulas that you used.

5. Points \( F(-3, 4) \), \( G(2, 2) \), \( H(2, -4) \), and \( J(-3, -2) \) are the vertices of parallelogram \( FGHJ \).
   a. Draw the parallelogram on the grid.
   b. Determine the perimeter of the parallelogram to the nearest tenth.
   c. What is the area of the parallelogram?
Lesson 30-1
Areas of Rectangles and Parallelograms

Lisa realizes that she can apply what she has learned to determine the measures of tabletops that are formed using a combination of shapes.

6. A diagram of a custom tabletop is shown. The tabletop is a composite figure that consists of a rectangle and parallelogram. Lisa needs to calculate the cost of the tabletop.

   ![Diagram of a custom tabletop]

   a. What additional dimension(s) does Lisa need to know in order to find the area of the tabletop? How can she find this? Be sure to use properties or theorems to justify your answer.

   b. What is the cost of the tabletop? Show your work.

Check Your Understanding

7. Find the area of the composite figure shown.

8. Determine the area and perimeter of a rectangle with length 12 cm and diagonal length 13 cm.

9. Determine the area and perimeter of the parallelograms shown. Round the answers to the nearest tenth.

   a. 
   
   b. 

MATH TERMS

A composite figure is made up of two or more simpler shapes, such as triangles, rectangles, or parallelograms. The word composite means made up of distinct parts. For example, in geology, a composite volcano is made up of alternating layers of lava and rocks.
LESSON 30-1 PRACTICE

Lisa was given the list of all of the shapes and necessary dimensions of some tabletops. In Items 10–12, determine the perimeter and area of each tabletop.

10. Table 1

11. Table 2

12. Table 3: Figure QRST with vertices Q(−1, 4), R(−1, 7), S(4, 7), and T(4, 4). Each unit of the coordinate plane represents one foot.

13. Reason quantitatively. Given that the charge for each tabletop is $8.50 per square foot, create an itemized invoice to show the cost of each tabletop from Items 10–12 and the total cost.

14. The area of the parallelogram shown is 45 square units. Find the height and base of the parallelogram.

15. Explain how to use a composite figure to estimate the area of the irregular figure shown on the coordinate plane.
**Learning Targets:**

- Solve problems using the areas of triangles and composite figures.
- Use coordinates to compute perimeters and areas of figures.

**SUGGESTED LEARNING STRATEGIES:** Marking the Text, Group Presentation, Identify a Subtask, Use Manipulatives, Quickwrite

Lisa asks Rashid to help her understand how to determine the area of triangular-shaped tabletops. Lisa and Rashid start the investigation with right and isosceles triangles.

1. Use the given triangles below to create quadrilaterals. Using what you know about these quadrilaterals, explain why the formula for the area of a triangle is $\text{Area} = \frac{1}{2} \cdot \text{base} \cdot \text{height}$.

    a. 
    
    b. 

2. **Reason abstractly.** Compare and contrast the steps required to find the area of a right triangle and an isosceles triangle whose side lengths are given.

3. Find the area and subsequent charge for each tabletop below.

    a. 
    
    b. 
    
    c. 

**MATH TIP**

An isosceles triangle has two congruent sides and two congruent angles.
4. Reason abstractly. Rashid draws several tabletops in the shape of an equilateral triangle, each having different side lengths. Derive an area formula that will work for all equilateral triangles with side length $s$.

5. Use the formula created in Item 4 to find the area of an equilateral triangle with side length 6 feet. Compare your answer to the area you found in Item 3c.

Check Your Understanding

6. Find the area of the tabletop shown on the coordinate plane.
   a. Name the shapes that make up the tabletop.
   b. Critique the reasoning of others. Jalen and Bree each used a different method to determine the area of the tabletop. Jalen found the sum of the areas of each of the four figures. Bree drew auxiliary lines to create a large rectangle, and then subtracted the area of the two triangular regions not in the tabletop from the large rectangle. Whose method is correct? Explain your reasoning.
   c. Compute the area of the tabletop.
   d. What are some benefits of working with composite figures on a coordinate plane?

7. Determine the area of triangle $ABC$. Leave the answer in radical form.

8. Explain how to determine the height of a right triangle versus the height of an equilateral triangle, given the side lengths of the triangles.
Next, Rashid shows Lisa how to determine the area of a triangular-shaped tabletop given two side lengths and the measure of the included angle.

9. **Make use of structure.** Complete the proof to discover an area formula that can be used when given a triangle with two side lengths and the measure of the included angle.

![Diagram of a triangle with labels A, B, C, a, b, c, and central angle A]

- **a.** Write the formula for the area of a triangle.
- **b.** Draw and label a perpendicular line, $h$, from point $B$ to $AC$. This represents the height of the triangle.
- **c.** Complete to find $\sin A$: $\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{b}{c}$
- **d.** Solve for $h$: $h = \frac{b}{2}$
- **e.** Substitute the equivalent expression you got for $h$ (from part d) into the formula for area of a triangle (from part a).

$$\text{Area} = \frac{1}{2}bh$$

$$\text{Area} = \frac{1}{2}b$$

10. The diagram shown represents tabletop $ABC$. The tabletop has side lengths 5 ft and 7 ft. The angle between the two sides, $\angle A$, has a measure of $49^\circ$. Compute the area of the triangular tabletop, to the nearest tenth.

![Diagram of a triangle with labels A, B, C, a, 5 ft, and 7 ft]
LESSON 30-2 PRACTICE

For Items 13–16, compute the area of each triangular-shaped tabletop, to the nearest tenth.

13. Equilateral triangle with 8.5 ft sides
14. Triangle ABC with vertices at (2, 1), (8, 1), and (6, 4)
15. Base measure of 15 ft, another side length of 12 ft, and an included angle of 30°
16. Isosceles triangle with sides of 4 ft, 7 ft, and 7 ft
17. Make sense of problems. Determine the perimeter and area of the tabletop shown on the coordinate plane below.
Learning Targets:
• Solve problems using the areas of rhombuses, trapezoids, and composite figures.
• Solve problems involving density.

SUGGESTED LEARNING STRATEGIES: Vocabulary Organizer, Think-Pair-Share, Create Representations, Quickwrite

Lisa begins to explore other tabletop templates. She uses what she has learned about rectangles, triangles, and parallelograms to investigate the areas of other polygons.

1. Included in the tabletop templates is the rhombus shown.

\[ \text{Given: } AB = 5', BC = 5', CD = 5', DA = 5', \text{ and } AC = 6'. \]

a. List the properties of a rhombus that relate to the diagonals.

b. Find the length of diagonal \( AC \).

c. Apply the formula for the area of the triangles formed by the diagonals to find the area of the rhombus.

d. Derive a formula for the area of a rhombus with diagonal lengths \( d_1 \) and \( d_2 \).
The area of a trapezoid with base lengths $b_1$ and $b_2$ and height $h$ can be derived by applying what you have already learned about the area of a triangle.

2. Use the figure shown to derive a formula for the area of a trapezoid. Explain how you arrived at your answer.

![Diagram of a trapezoid with bases $b_1$ and $b_2$ and height $h$.]

3. Critique the reasoning of others. Lisa states, “The area of a trapezoid is equal to the length of its median times its height.” Is she correct? Why?

4. Find the area of each of the trapezoids shown below.

   a. 
   
   b.
Lesson 30-3  
Areas of Rhombuses and Trapezoids

To determine the total cost of a tabletop, Lisa needs to also consider the cost of shipping. The greater the mass of an object, the greater the shipping cost. The mass of the tabletop is dependent on the density of the material used in its construction.

5. A Cut Above is making the tabletop shown in Item 4a out of two different types of wood: western red cedar and maple. The volume of each tabletop is 4 ft$^3$. The density of western red cedar is 23 lb/ft$^3$, and the density of maple is 45 lb/ft$^3$. If the cost of shipping the tabletops is $0.50 per pound, which tabletop costs more to ship? How much more?

6. Find the area of each trapezoid.
   a. 
   \[
   \begin{array}{c}
   \text{12 in.} \\
   45^\circ \\
   20 \text{ in.} \\
   45^\circ
   \end{array}
   \]
   b. 
   \[
   \begin{array}{c}
   \text{26 in.} \\
   60^\circ \\
   14 \text{ in.}
   \end{array}
   \]

7. The density of bamboo is about 20 lb/ft$^3$. What is the mass of a bamboo tabletop that has a volume of 30 ft$^3$?
**LESSON 30-3 PRACTICE**

For Items 8–11, find the area, given the measures of each figure.

8. Isosceles trapezoid with base lengths 2 ft and 5 ft and leg lengths 2.5 ft

9. Rhombus with diagonal lengths 6 in. and 10 in.

10. 

```
           16 in.
          /     \\
  60°     /     \\
             /     \\
          30 in.
```

11. The area of an isosceles trapezoid is 54 square cm. The perimeter is 32 cm. If a leg is 7 cm long, find the height of the trapezoid.

12. **Attend to precision.** The front face of a manufactured home is shown in the diagram below.

```
          21 ft
         /     \\
  16 ft     /     \\
         /     \\
          18 ft
```

a. Compute the area.

b. If the density of the material from which the home is made is 38 lb/ft³, and the volume of the home is 856 ft³, what is the mass?
ACTIVITY 30 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 30-1
For Items 1–5, find the area of each shape described or shown.

1. Rectangle with base length 10 cm and diagonal length 15 cm
2. Equilateral triangle with perimeter 30 in.
3. Square inscribed in a circle with radius 16 mm
4. Figure $TUVW$ with vertices $T(-3, 6), U(-3, 7), V(0, 7), W(0, 6)$
5. What is the perimeter of Figure $ABCD$ with vertices at $A(2, 3), B(5, 3), C(5, -3), D(2, -3)$? What is the area?

Lesson 30-2
For Items 7–8, find the area of each shape described or shown.

7. Equilateral triangle with height 6 cm
8. $30^\circ-60^\circ-90^\circ$ triangle with hypotenuse length 14 in.
9. Compute the perimeter and area of the figure shown, to the nearest tenth.
Use Figure $ABCDE$ on the coordinate plane for Items 10–11.

10. Which equation can be used to determine $CD$?
   A. $CD = \sqrt{(5 - 0)^2 + (5 - 0)^2}$
   B. $CD = \sqrt{(5 - 5)^2 + (0 - 0)^2}$
   C. $CD = \sqrt{(5 - 0)^2 - (5 - 0)^2}$
   D. $CD = \sqrt{(5 - 5)^2 + (5 + 5)^2}$

11. What is the area of Figure $ABCDE$?
   A. 25 ft$^2$
   B. 22.5 ft$^2$
   C. 32.5 ft$^2$
   D. 56 ft$^2$

12. The base and height of a triangular-shaped countertop have a 5 : 2 ratio. Given that the area of the countertop is 320 in.$^2$, what are the measures of the base and height?

Lesson 30-3

For Items 13–14, find the area of each shape described or shown.

13. The lengths of the diagonals of a rhombus are 25 cm and 40 cm.

14. Trapezoid with height of 8 in. and bases of 10 in. and 15 in.

15. The length of one diagonal of a rhombus is 10 in, and its area is 70 in.$^2$. Find the length of the other diagonal.

16. The pillar shown has a thickness of 0.5 ft. To find its volume, the thickness is multiplied by its area. If the pillar has a mass of 5950 lb, from which material is the pillar most likely made?

   A. walnut: density = 35 lb/ft$^3$
   B. pine: density = 30 lb/ft$^3$
   C. poplar: density = 22 lb/ft$^3$
   D. balsa: density = 9 lb/ft$^3$

MATHEMATICAL PRACTICES

Construct Viable Arguments and Critique the Reasoning of Others

17. How could you use the formula for area of a trapezoid to find the area of a parallelogram?
Regular Polygons

Plenty of Polygons

Lesson 31-1 Sum of the Measures of the Interior Angles of a Polygon

Learning Targets:

• Develop a formula for the sum of the measures of the interior angles of a polygon.
• Determine the sum of the measures of the interior angles of a polygon.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Create Representations, Look for a Pattern, Self Revision/Peer Revision

A **polygon** is a closed geometric figure with sides formed by three or more coplanar segments that intersect exactly two other segments, one at each endpoint. The angles formed inside the polygon are **interior angles**. Although it is difficult to measure lengths and angles exactly, tools such as rulers and protractors allow you to measure with reasonable accuracy.

Work with your group and use the polygons on the next two pages.

1. **Use appropriate tools strategically.** Measure, as precisely as possible, each interior angle, and record your results below. Complete the table by calculating the indicated sums.

<table>
<thead>
<tr>
<th></th>
<th>1st Angle</th>
<th>2nd Angle</th>
<th>3rd Angle</th>
<th>4th Angle</th>
<th>5th Angle</th>
<th>6th Angle</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quadrilateral</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pentagon</td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Compare your results in the table with those of other groups in your class. What similarities do you notice?

3. Write a conjecture about the sum of the measures of the interior angles of each of the polygons.
   - Triangle:
   - Quadrilateral:
   - Pentagon:
ACTIVITY 31
continued

Lesson 31-1
Sum of the Measures of the Interior Angles of a Polygon

My Notes
Knowing that the sum of the measures of the interior angles of a triangle is a constant and that the sum of the measures of nonoverlapping angles around a single point is always $360^\circ$, you can determine the sum of the measures of any polygon without measuring.

4. Use auxiliary segments to determine a way to predict and verify the exact sum of the angles of any quadrilateral and any pentagon. Describe your methods so that another group would be able to replicate your results for the pentagon.

5. Use the method you described in Item 4 to answer the following.
   a. Explain how to determine the sum of the measures of the interior angles of a hexagon.

   b. Explain how to determine the sum of the measures of the interior angles for any polygon.

6. Express regularity in repeated reasoning. Use the method described in your answer to Item 5 to complete the table below.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of Sides</th>
<th>Calculations</th>
<th>Sum of the Measures of the Interior Angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td>3</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pentagon</td>
<td>5</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hexagon</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heptagon</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Octagon</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nonagon</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Decagon</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Dodecagon</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n$-gon</td>
<td>$n$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Lesson 31-1
Sum of the Measures of the Interior Angles of a Polygon

7. Observe in Item 6 that as the number of sides increases by one, the sum of the angle measures also increases by a constant amount. What type of function models this behavior?

8. For the first six polygons in Item 6, plot the ordered pair (number of sides, sum of angle measures) on the axes below. Carefully choose and label your scale on each axis.

9. The data points you graphed above should appear collinear. Write an equation for the line determined by these points.

10. Make sense of problems. State the numerical value of the slope of the line in Item 9 and describe what the slope value tells about the relationship between the number of sides and the sum of the measures of the interior angles of a polygon. Use units in your description.

11. If the function \( S \) represents the sum of the measures of the interior angles as a function of the number of sides, \( n \), write an algebraic rule for \( S(n) \).

12. Compare how the algebraic rule in Item 11 is similar to the slope-intercept form of a linear equation.
13. Use \( S(n) \) to determine the value of \( S(7.5) \). For this value, explain the significance, if any, given that \( S(n) \) represents the sum of the angle measures of an \( n \)-sided polygon.

14. What is the domain of \( S(n) \)?

15. How can you tell by looking at the graph in Item 8 that the function is linear? Explain why the domain is restricted.

16. Determine the sum of the interior angles of a 20-sided polygon.

17. What is the measure of each interior angle of a regular polygon with 60 sides?

18. The sum of the interior angles of a polygon is \( 2340^\circ \). How many sides does the polygon have?

19. Does the rule in Item 11 hold true for various types of triangles? Create a way to prove the rule for a scalene triangle, a right triangle, and an isosceles triangle. Start by drawing one of the triangles, tearing it into three pieces, and rearranging the vertices.

20. Critique the reasoning of others. Denisha states that if you are given the sum of the interior angles of a polygon, there is no way to determine the number of sides of the polygon. Do you agree with Denisha? Justify your reasoning.

21. Use technology to create a hexagon. Label all angle measures but one. Exchange figures with a partner and find the missing angle measures in each hexagon using the rule you created in Item 11. Check answers using technology.

22. The angle measures of a stop sign are all the same. Determine the measure of each angle in a stop sign.
Learning Targets:
- Develop a formula for the measure of each interior angle of a regular polygon.
- Determine the measure of the exterior angles of a polygon.

**SUGGESTED LEARNING STRATEGIES:** Think-Pair-Share, Create Representations, Look for a Pattern, Quickwrite

A regular polygon is both equilateral and equiangular. This means that each interior angle of a regular polygon has the same angle measure.

1. Complete the table to determine the angle measure of each interior angle for each regular polygon listed.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of Sides</th>
<th>Sum of the Measures of the Interior Angles (degrees)</th>
<th>Measures of Each Interior Angle (degrees)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangle</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Quadrilateral</td>
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<td></td>
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<tr>
<td>Pentagon</td>
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<td>Hexagon</td>
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<tr>
<td>Decagon</td>
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<td>Dodecagon</td>
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<td></td>
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</tr>
<tr>
<td>n-gon</td>
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<td></td>
<td></td>
</tr>
</tbody>
</table>

Equiangular means that all angles of a polygon are congruent. Equilateral means all sides of a polygon are congruent.
2. For each regular polygon listed in the table in Item 1, plot the ordered pair (number of sides, measure of each interior angle) on the axes below. Carefully choose and label your scale on each axis.

3. **Make use of structure.** The points plotted in Item 2 should not appear collinear. Explain how that conclusion could have been drawn from the data alone.

4. If the function $E$ represents the measure of each interior angle as a function of the number of sides, $n$, write an algebraic rule for $E(n)$.

5. As $n$ becomes greater, what appears to be happening to the measure of each angle? What causes this behavior?
Exterior angles of a polygon also have a relationship. An **exterior angle** of a polygon is the angle formed between one side of a polygon and an extension of an adjacent side.

6. Use geometry software to draw a **convex** pentagon. Extend each side.

7. **Use appropriate tools strategically.** Use the software to measure the exterior angles at each vertex. To do this you will need to mark a point on each ray.

8. Find the sum of the exterior angles from Item 7.

9. Manipulate the polygon to form a convex hexagon. Measure each exterior angle and find the sum of the measures.

10. Manipulate the polygon to form a convex heptagon. Measure each exterior angle and find the sum of the measures.

11. Make a conjecture about the sum of the exterior angles of a polygon with $n$ sides.

12. Predict the sum of the measure of the interior angle of a pentagon and one of its exterior angles. On what basis do you make your prediction? Use the software to measure and confirm your prediction.
Lesson 31-2
Regular Polygons and Exterior Angles

14. What is the sum of the interior angles of a regular 24-gon? What is the measure of each interior angle?

15. How does the sum of the measure of the exterior angles of a convex polygon differ if the polygon is a regular polygon? Explain.

16. A convex pentagon has exterior angles that measure $70^\circ$, $105^\circ$, $28^\circ$, and $32^\circ$. What is the measure of the fifth exterior angle?

17. Each exterior angle of a regular polygon measures $24^\circ$. How many sides does the polygon have?

Check Your Understanding

18. Find the sum of the measures of the interior angles of each regular polygon.
   a. 15-gon  
   b. 18-gon

19. The expression $(a - 4)^\circ$ represents the measure of an interior angle of a regular 20-gon. What is the value of $a$ in the expression?

20. Find the measure of each interior angle of each regular polygon.
   a. 15-gon  
   b. 18-gon

21. Find the sum of the measures of the exterior angles of each polygon.
   a. 15-gon  
   b. 18-gon

22. The expression $(n + 3)^\circ$ represents the measure of an exterior angle of a regular 18-gon. What is the value of $n$ in the expression?

23. Construct viable arguments. Emilio said that he drew a regular polygon using a protractor and measured one of its interior angles as $100^\circ$. Explain, using what you know about exterior angles, why this cannot be true.

24. Use graphing software to draw three quadrilaterals: a square, a rectangle, and a nonequiangular quadrilateral.
   a. Find the measure of each interior angle of each polygon.
   b. Prove that the sum of the measures of the interior angles of each quadrilateral is the same.
   c. Prove that the sum of the measures of the exterior angles of each quadrilateral is $360^\circ$.
Lesson 31-3
Area and Perimeter of Regular Polygons

Learning Targets:
- Develop a formula for the area of a regular polygon.
- Solve problems using the perimeter and area of regular polygons.

SUGGESTED LEARNING STRATEGIES: Think-Pair-Share, Quickwrite, Create Representations

Recall that a regular polygon is equilateral and equiangular. An **apothem** of a regular polygon is a perpendicular segment from a midpoint of a side of a regular polygon to the center of the circle circumscribed about the polygon. The apothem of a regular hexagon of side \( s \) is indicated by \( a \) in the diagram.

\[
\begin{align*}
\text{S} & \quad \text{d} \\
\end{align*}
\]

1. Consider the tabletop template of the regular hexagon shown below.

\[
\begin{align*}
\text{a.} & \quad \text{Draw all radii from the center of the circumscribed circle to each vertex of the hexagon.} \\
\text{b.} & \quad \text{What is the measure of each of the central angles formed by the radii? Explain how you arrived at your answer.} \\
\text{c.} & \quad \text{Classify the triangles formed by the radii by their side length. Measure to verify your answer.} \\
\text{d.} & \quad \text{How is the apothem of the hexagon related to the triangles formed by the radii?} \\
\text{e.} & \quad \text{Find the area of a regular hexagon with side length 4 ft. Show the calculations that lead to your answer.}
\end{align*}
\]
Now you can use what you discovered in Item 1 to generalize the formula to determine the area of any regular polygon.

2. **Express regularity in repeated reasoning.** Refer to the diagrams of the regular polygons shown.

![Diagram of regular polygons]

a. How many triangles are formed when the radii are drawn from the center of the polygon to each of the vertices of the polygon with $n$ sides?

b. What is the measure of each of the central angles formed by the radii of the $n$-gon? Explain how you arrived at your answer.

c. Classify the triangles formed by the radii by their side length. Verify your answer.

d. Write an expression that finds the area of one triangle in an $n$-gon, using $a$ to represent the apothem and $s$ to represent a side of the $n$-gon. Then use the expression to write a formula that finds the area of the entire $n$-gon, where $n$ represents the number of sides of the $n$-gon.

e. Explain how to determine the perimeter of a polygon, $P$.

f. Write a formula that can be used to calculate the area of a regular $n$-gon with apothem length $a$ and perimeter $P$, by substituting $P$ in the formula you wrote in part d.
Lesson 31-3
Area and Perimeter of Regular Polygons

Example A
Calculate the area of a regular octagon with side length 8 inches.

Step 1: Find the apothem.

Draw the apothem, \( QS \).
The length of \( QS \) is \( a \).

Since the polygon is a regular octagon,
\[ m\angle PQR = \frac{360^\circ}{8} = 45^\circ. \]
So, \( m\angle SQR = 22.5^\circ. \)

Use trigonometry to find \( a \):
\[ \frac{4}{a} = \tan 22.5^\circ, \text{ so } a \approx 9.7 \text{ in.} \]

Step 2: Find the area.

\[ A = \frac{1}{2} \cdot a \cdot P \]
\[ = \frac{1}{2} (9.7 \text{ in.})(8 \text{ in.}) \]
\[ = 310.4 \text{ in.}^2 \]

Solution: The area is approximately 310.4 square inches.

Try These A

a. Calculate the area of a regular pentagon with side length 12 m.

b. Find the area of a regular hexagon with 18 cm sides.

c. Reason quantitatively. Find the perimeter and area of regular polygon \( ABCD \) with vertices \( A(7, 8), B(9, -1), C(0, -3), \) and \( D(-2, 6). \)
Check Your Understanding

3. The properties of what type of triangle can be used to determine the apothem in a regular hexagon?

4. Write a formula that can be used to determine the perimeter of a regular $n$-gon.

5. Find the area of a regular octagon with 7-inch sides.

6. Seth states that the radius of a regular polygon is never less than its apothem. Do you agree? If so, provide justification. If not, provide a counterexample.

LESSON 31-3 PRACTICE

7. Determine the perimeter and area of the regular dodecagon.

8. Make sense of problems. A manufacturing company is making a design for an octagonal hazard sign. Each side of the sign measures 40 in. What is the area of the sign?

9. Find the area of regular polygon $QRST$ with vertices at $Q(1, 3)$, $R(5, 0)$, $S(2, -4)$, and $T(-2, -1)$.

10. Find the area of a regular pentagon with a radius of 8 cm.
ACTIVITY 31 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 31-1
Determine the missing angle measure for each polygon.

1. \[ \begin{align*}
\text{108°} & \quad \text{140°} \\
100° & \quad k
\end{align*} \]

2. \[ \begin{align*}
18° & \quad 94° \\
b & \quad \text{b}
\end{align*} \]

3. What is the value of \( x \)?

\[ \begin{align*}
88° & \quad (13x - 3)° \\
(7x + 15)° & \quad 80°
\end{align*} \]

4. If the sum of the measures of the interior angles of a polygon is 2700°, how many sides does it have?
   - A. 11
   - B. 15
   - C. 17
   - D. 29

5. If the measure of each interior angle of a polygon is 150°, how many sides does it have?
   - A. 10
   - B. 12
   - C. 14
   - D. 16

6. Given \( \triangle APT \) with angle measures as labeled, solve for \( x \) and calculate the measures of angles \( P \), \( A \), and \( T \).
Lesson 31-2

7. What is the measure of each angle in the polygon shown?

For Items 8–10, draw polygons to satisfy the given conditions.

8. An equiangular quadrilateral that is not equilateral
9. An equilateral hexagon that is not equiangular
10. A regular polygon with an angle measuring 135°
11. What is the difference in the sum of the exterior angles measures of a regular nonagon and a regular dodecagon?
   A. 0°
   B. 90°
   C. 180°
   D. 360°

Lesson 31-3

12. A regular pentagon has sides of 4 ft. What are the perimeter and area of the pentagon?
13. Find the area of an equilateral triangle with apothem of 6 cm.
14. Find the area of a regular hexagon inscribed in a circle with a radius of 6 ft.
15. A regular triangle and a regular quadrilateral share a common side. Write a ratio of the area of the rectangle to the area of the triangle.
16. A regular octagon is plotted on the coordinate plane. The distance between each vertex is 2 units. Determine the area of the octagon.
17. What is the radius of a 1092 ft² regular heptagon?

MATHEMATICAL PRACTICES
Construct Viable Arguments and Critique the Reasoning of Others

18. a. As the number of sides of a regular polygon increases, what happens to the shape of the polygon?
   b. As a regular polygon gets smaller and smaller, what happens to the exterior angles? How does this help prove the conjecture that the sum of the exterior angles of a polygon is 360°?
Lance owns The Flyright Company. His company specializes in making parachutes and skydiving equipment. After returning from a tour of Timberlake Gardens, he was inspired to create a garden in the circular drive in front of his office building. Lance plans to hire a landscape architect to design his garden. The architect needs Lance to provide the area and circumference of the proposed garden so the architect can estimate a budget.

Lance relies on the measures of some familiar objects to help him understand the measures of a circle.

<table>
<thead>
<tr>
<th>Object</th>
<th>Distance Around the Object, $C$</th>
<th>Diameter, $d$</th>
<th>$\frac{C}{d}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Coin</td>
<td>7.85 cm</td>
<td>2.5 cm</td>
<td>3.14</td>
</tr>
<tr>
<td>Clock</td>
<td>40.84 mm</td>
<td>13 mm</td>
<td>3.14</td>
</tr>
<tr>
<td>Frisbee</td>
<td>62.83 in.</td>
<td>20 in.</td>
<td>3.14</td>
</tr>
</tbody>
</table>

1. Complete the table to find the ratio of $C : d$.

2. Make a conjecture about the relationship between the circumference and the diameter of circular objects.

The **circumference** of a circle is the distance around the circle. Circumference is measured in linear units. The ratio of the circumference to the diameter of any circle is designated by the Greek letter $\pi$ (pi).

3. Write a formula, in terms of diameter, $d$, that can be used to determine the circumference of a circle, $C$.

4. Write a formula, in terms of the radius, $r$, that can be used to determine the circumference of a circle, $C$.

5. What information does the circumference provide about the garden Lance wants to create?

The first known calculation of $\pi$ (pi) was done by Archimedes of Syracuse (287–212 BCE), one of the greatest mathematicians of the ancient world. Archimedes approximated the area of a circle using the Pythagorean Theorem to find the areas of two regular polygons: the polygon **inscribed within the circle** and the polygon **within which the circle was circumscribed**. Since the actual area of the circle lies between the areas of the inscribed and circumscribed polygons, the areas of the polygons gave upper and lower bounds for the area of the circle. Archimedes knew that he had not found the value of pi but only an approximation within those limits. In this way, Archimedes showed that $\pi$ is between $3 \frac{10}{71}$ and $3 \frac{1}{7}$. 
Lance uses a compass and paper folding to develop the formula for the area of a circle.

6. **Use appropriate tools strategically.** Use a compass to draw a large circle. Use scissors to cut the circle out.

7. Divide the circle into 16 equal **sectors**.

8. Cut the sectors out and arrange the 16 pieces to form a parallelogram.

9. Write the formula for the area of a parallelogram, in terms of base $b$ and height $h$.

10. Explain how the height of the parallelogram is related to the radius of the circle.

11. Explain how the base of the parallelogram is related to the circumference of the circle.

12. How does the area of the parallelogram compare to the area of the original circle?

13. **Construct viable arguments.** Use your knowledge of area of parallelograms to prove the area formula of a circle.

14. What information does the area provide about the garden Lance wants to create?
The layout of The Flyright Company's building and parking lot is shown below.

The dimensions of the grassy area of the parking lot are 32 ft by 27 ft.

15. What are the radius, circumference, and area of the largest circle that will fit in the grassy area? Justify your answer.

Even though larger circular gardens will fit in the grassy area, Lance decides that he would like the garden to have a diameter of 20 feet.

16. The landscape architect recommends surrounding the circular garden with decorative edging. The edging is sold in 12-foot sections that can bend into curves. How many sections of the edging will need to be purchased to surround the garden? Justify your answer.

17. To begin building the garden, the landscape architect needs to purchase soil. To maintain a depth of one foot throughout the garden, each bag can cover 3.5 square feet of the circle. How many bags of soil need to be purchased? Show the calculations that lead to your answer.

18. Lance and the architect are discussing the possibility of installing a sidewalk around the outside of the garden, as shown in the diagram. Determine the area of the sidewalk.

19. Critique the reasoning of others. Darnell claims that the circumference of a circular garden with radius 12 meters is greater than the perimeter of a square garden with side 18 meters. Do you agree? Explain your reasoning.
Lesson 32-1
Circumference and Area of a Circle

Check Your Understanding

20. Explain why the units of measure for the circumference of a circle are linear, and those of the area of a circle are in square units.
21. Express the radius of a circle in terms of its circumference.
22. Express the area of a circle in terms of its circumference.
23. If Lance decides to increase the width of the walkway to 3 feet and keep the outer radius at 12 feet, what will happen to the area of the garden?

LESSON 32-1 PRACTICE

24. A circular rotating sprinkler sprays water over a distance of 9 feet. What is the area of the circular region covered by the sprinkler?
25. Reason quantitatively. What is the greatest circumference of the circle that can be inscribed in the square below?

\[
\text{10 cm}
\]

26. Suppose a landscaper pushes a wheelbarrow so that the wheel averages 25 revolutions per minute. If the wheel has a diameter of 18 inches, how many feet does the wheelbarrow travel each minute?
27. Construct viable arguments. Can the area of a circle ever equal the circumference of a circle? Support your answer with proof.
Lesson 32-2
Sectors and Arcs

Learning Targets:
• Develop and apply a formula for the area of a sector.
• Develop and apply a formula for arc length.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Create Representations, Quickwrite

Lance told the landscape architect that he would like the garden to resemble a colorful parachute with different flowers in alternating areas of the garden. The landscaper developed the sketch shown below.

1. Model with mathematics. The circular garden is divided into 8 equal parts.
   a. What portion of the total area of the circle is each part?

   b. Recall that Lance would like the diameter of the circular garden to be 20 feet. Determine the area of each sector of the garden. Show the calculations that led to your answer.

   c. What is the measure of the central angle for the sector from part a?

   d. Write the fraction that is equivalent to your answer in part a and that has a denominator of 360. Explain the meaning of both the numerator and denominator using circle terminology.

2. Write an equation that will allow Lance to calculate the area of a sector with a central angle of $n^\circ$ and radius $r$. 

Activity 32 • Length and Area of Circles

You will work with sectors of circles when you study polar equations in calculus.
3. Determine the arc length of each sector of the garden.

4. Write an equation that will allow Lance to calculate the arc length of a sector with a central angle of $n^\circ$ and radius $r$.

5. Compare and contrast the arc length of a sector and the area of a sector.

6. The architect submits his estimate to Lance. Lance notices that the architect has used the following formulas to calculate the area of each sector and length of each arc in the garden.

\[
\frac{\text{degree measure of the central angle}}{360^\circ} = \frac{\text{area of sector}}{\text{area of the circle}}
\]

\[
\frac{\text{degree measure of the central angle}}{360^\circ} = \frac{\text{arc length of a sector}}{\text{circumference of the circle}}
\]

Are the equations you developed in Item 2 and Item 4 equivalent to the architect’s equations? Explain your answer.
Lesson 32-2
Sectors and Arcs

Check Your Understanding

7. If a circle has six equal sectors, explain how to determine the measure of the central angle of each sector.

8. Name the term for the sum of all the arc lengths in a circle.

9. Critique the reasoning of others. Juliet claims that the length of an arc is the same thing as the measure of an arc. Do you agree? Explain why.

10. The length of an arc in a circle with diameter 10 inches is $1.25\pi$ inches. Determine the measure of the arc.

11. The measure of the central angle of a sector is $36^\circ$ and the radius of the circle is 5 inches. What is the area of the sector?

LESSON 32-2 PRACTICE

12. The length of an arc in a circle with radius 8 inches is $3.2\pi$ inches. Determine the measure of the arc.

13. The measure of the central angle of a sector is $60^\circ$ and the area of the sector is $6\pi$ inches$^2$. Calculate the radius of the circle.

14. Attend to precision. Use circle $B$ below to find the following measures.

![Diagram of circle with angle and radius labeled]

a. area of circle $B$

b. area of shaded sector of circle $B$

c. circumference of circle $B$

d. length of minor $AC$
Lance likes the design ideas generated by the landscape architect. The architect asks Lance if he will consider making the garden larger in order to include a circular fountain in the center of the garden. Lance is concerned that it will take a long time to update the current sketches and budgets to accommodate the larger garden. The architect assures him that similarity properties apply to all circles and that the changes will be quick and easy. Lance asks, "Are you sure that all circles are similar?"

The architect draws circle $O$ with radius $r$ to represent the original design of the garden, and circle $P$ with radius $s$ to represent the sketch of the new garden.

1. **Reason abstractly.** If it is true that any two circles are similar, can one circle be mapped onto the other? Explain your answer.

2. The architect maps the center of circle $O$ onto the center of circle $P$, as shown in the diagram.

   a. What transformations are performed?

   b. If $O'$ is the image of circle $O$, then identify the location of $O'$. 

**Math Terms**

*Concentric circles* share the same center.
Lesson 32-3
Circles and Similarity

3. **Make use of structure.** What is the scale factor between the circles?

4. Two circles share the same center. The radius of one circle is 12, and the radius of the other circle is 8.
   a. Describe how the smaller circle was dilated to generate the larger circle.
   
   b. Compare the ratios of the circumferences of the two circles.
   
   c. Compare the ratios of the two areas of the two circles.
   
   d. Compare the ratios you found in parts b and c to the scale factor. Describe any patterns you notice.

5. Circle S is located at (0, 0). It has a radius of 4 units. Circle T is located at (2, 5). It has a radius of 10 units. Describe the sequence of transformations that proves that circle S is similar to circle T.

6. Circle Q, with radius of 18 feet, is reduced by 60 percent. What is the radius of the new circle?
To finalize the design of the garden, the landscape architect is using a landscaping software program. To operate it correctly, the architect has to enter some of the angles in radians and some of the angles in degrees. Recall the formula for the arc length $s$ of an arc with measure $m^\circ$ and radius $r$: $s = \frac{m^\circ}{360} (2\pi r)$. This formula can be applied to concentric circles.

7. Given the three concentric circles below, with radius 1 unit, 2 units, and 3 units, do the following.
   a. Calculate the arc lengths of each of the three arcs formed by the $60^\circ$ angle.

   ![Diagram of three concentric circles with a $60^\circ$ angle](image)

   b. Complete the following table for the three circles. Do not use an approximation for $\pi$.

<table>
<thead>
<tr>
<th>Circle</th>
<th>Measure of Central Angle</th>
<th>Radius</th>
<th>Arc Length</th>
<th>Arc Length: Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

8. **Express regularity in repeated reasoning.** What pattern do you notice in the table?
Lesson 32-3
Circles and Similarity

The architect points out that for a given fixed angle measure \( m \), the length of the arc cut off by the central angle is proportional to the length of the radius. This constant of proportionality, \( \frac{m^\circ}{360^\circ} \cdot 2\pi = \frac{m^\circ}{180^\circ} \cdot \pi \), defines the **radian measure** of the angle.

**Example A**
Converting degrees to radians.
Convert 150° to radians.
Substitute \( m = 150 \) into the expression \( \frac{m^\circ}{180^\circ} \cdot \pi \).
\[
\frac{m^\circ}{180^\circ} \cdot \pi = \frac{150^\circ}{180^\circ} \cdot \pi = \frac{5\pi}{6}
\]
So, 150° = \( \frac{5\pi}{6} \) radians.

**Example B**
Converting radians to degrees.
Convert \( \frac{\pi}{5} \) radians to degrees.
Multiply radians by \( \frac{180^\circ}{\pi} \).
\[
\frac{180^\circ}{\pi} \left( \frac{\pi}{5} \right) = 36^\circ
\]
So, \( \frac{\pi}{5} \) radians = 36°.

**Try These A-B**

a. Complete: 20° = _________ radians
b. Complete: 270° = _________ radians
c. Complete: \( \frac{2\pi}{3} \) radians = _________°
d. Complete: \( \frac{\pi}{9} \) radians = _________°

**CONNECT TO SCIENCE**

Units are an important part of scientific values and calculations. Scientists use one system of measurement—the SI system—to make it easier to communicate among themselves. The radian is the SI unit for angular measurement. Scientists often need to convert units using conversion ratios, as you may have done in your science classes.
11. Make sense of problems. Circle $O$ is located at $(2, 1)$. It has a radius of 4 units. Circle $P$ is located at $(-1, -1)$. It has a radius of 2 units.
   a. Are the circles congruent? How do you know?
   b. Can you prove that circle $O$ and circle $P$ are similar using only one similarity transformation? Explain.
   c. How can you prove that circle $O$ and circle $P$ are similar using more than one similarity transformation?

12. What degree measure is equivalent to the following radian measures?
   a. $\frac{\pi}{2}$
   b. $\frac{\pi}{4}$

13. What radian measure is equivalent to the following angle measures?
   a. $75^\circ$
   b. $150^\circ$
   c. $300^\circ$
ACTIVITY 32 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 32-1
1. A circular ceramic plate has a circumference of $12\pi$ inches. What is the area of the plate?
2. What is the approximate largest circumference of a circular pond that could fit within a square walkway of sides 30 meters?
   A. 47 m
   B. 60 m
   C. 94 m
   D. 120 m
3. Levi boards a Ferris wheel with a diameter of 100 feet. What is the approximate distance, to the nearest foot, that Levi travels in six revolutions of the wheel?

Lesson 32-2
Refer to the circle below for Items 7 and 8.

4. A parallelogram has an area of 324 square feet. A circle has the same area. What is the approximate circumference of the circle?
   A. 10.2 ft
   B. 18.2 ft
   C. 31.8 ft
   D. 63.8 ft
5. Which of the following is equivalent to the ratio of the circumference to the radius of a circle?
   A. $\pi$
   B. $\frac{\pi}{2}$
   C. $2\pi$
   D. $4\pi$
6. The circumference of the thin metal band that protects the circular glass face of Dale’s watch is $16\pi$ cm. What is the area of the glass face?

7. Determine the area of the shaded region.
8. Determine the length of minor $\overline{AB}$. 
9. The measure of the arc of a sector is 72° and the area of the sector is $5\pi$ in.$^2$. What is the radius of the circle?
   A. 5 in.
   B. 9 in.
   C. 10 in.
   D. 25 in.

10. Is it possible for an arc with a central angle of 30° in one circle to have a greater arc length than an arc with a central angle of 150° in another circle? Justify your reasoning.

11. Find the area, to the nearest tenth, of one-quarter of a circular mirror with diameter 8 meters.

Lesson 32-3

12. Circle $Q$ is located at $(-5, 2)$. It has a radius of 5 units. Circle $O$ is located at $(5, 2)$. It has a radius of 2 units.
   a. Describe the sequence of transformations that maps circle $Q$ onto circle $O$ to prove that the circles are similar.
   b. Describe another sequence of transformations that can be used to prove the circles are similar.

13. Circle $S$, with radius of 14 cm, is dilated using the scale factor $4 : 5$. What is the radius of the new circle?

14. Which of the following is equivalent to $120^\circ$?
   A. $\frac{1}{3}\pi$ radians
   B. $\frac{2}{3}\pi$ radians
   C. $2\pi$ radians
   D. $3\pi$ radians

15. What degree measure is equivalent to $\frac{7\pi}{4}$ radians?

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

16. If the area of a sector is one-tenth of the area of the circle, what is the central angle of the sector? Explain how you determined your answer.
Some students are building the stage set for an upcoming school play. As they finalize some of the backdrops, they need to determine the budget for the cost of materials. The paint costs $12.50 per gallon, and one gallon of paint covers 350 square feet. Reflecting tape costs $4.50 per roll, and one roll of tape covers 40 linear feet.

1. The wooden backdrop for the opening scene shows a city skyline. One side of the backdrop is to be painted black. Find the area of the skyline. Then determine the budget for black paint if the backdrop is to be painted with two coats of paint.

2. The second scene of the play takes place in a beehive. The backdrop is in the shape of a regular hexagon, as shown on the coordinate plane below. Each unit on the coordinate plane represents 5 feet.
   a. Determine the budget for painting one side of the backdrop with one coat of yellow paint.
   b. The perimeter of the hexagon is covered with reflecting tape. Determine the budget for the reflecting tape.
3. Part of the school’s stage is shaped like a sector of a circle, as shown below. The students do not yet know the exact dimensions of this part of the stage, but they want to identify as many relationships as possible so that they will be able to decorate this part of the stage when the dimensions are available.

\[ \frac{s}{a} = \frac{t}{a + b}. \]

a. Explain why \( \frac{s}{a} = \frac{t}{a + b} \).

b. The length of \( s \) is 12 feet, and the length of \( a \) is 10 feet. Determine the measure of \( \angle 1 \) in both degrees and radians.

<table>
<thead>
<tr>
<th>Scoring Guide</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mathematics Knowledge and Thinking (Items 1, 2, 3)</td>
<td>- Clear and accurate understanding of how to find the area and perimeter of plane figures, the proportional reasoning required to determine the budget for supplies, and arc length and conversions between degree and radian measures of angles</td>
<td>- A functional understanding of how to find the area and perimeter of plane figures, the proportional reasoning required to determine the budget for supplies, and arc length and conversions between degree and radian measures of angles</td>
<td>- Partial understanding of how to find the area and perimeter of plane figures, the proportional reasoning required to determine the budget for supplies, and arc length and conversions between degree and radian measures of angles</td>
<td>- Little or no understanding of how to find the area and perimeter of plane figures, the proportional reasoning required to determine the budget for supplies, and arc length and conversions between degree and radian measures of angles</td>
</tr>
<tr>
<td>Problem Solving (Items 1, 2, 3b)</td>
<td>- An appropriate and efficient strategy that results in correct answers</td>
<td>- A strategy that may include unnecessary steps but results in a correct answer</td>
<td>- A strategy that results in some incorrect answers</td>
<td>- No clear strategy when solving problems</td>
</tr>
<tr>
<td>Mathematical Modeling / Representations (Item 3a)</td>
<td>- Clear and accurate understanding of using arc length to create a proportion</td>
<td>- Mostly accurate understanding of using arc length to create a proportion</td>
<td>- Partial understanding of using arc length to create a proportion</td>
<td>- Little or no understanding of using arc length to create a proportion</td>
</tr>
<tr>
<td>Reasoning and Communication (Item 3a)</td>
<td>- Precise use of appropriate mathematics and language to justify ( \frac{s}{a} = \frac{t}{a + b} )</td>
<td>- Mostly correct use of appropriate mathematics and language to justify ( \frac{s}{a} = \frac{t}{a + b} )</td>
<td>- Misleading or confusing use of appropriate mathematics and language to justify ( \frac{s}{a} = \frac{t}{a + b} )</td>
<td>- Incomplete or inaccurate use of appropriate mathematics and language to justify ( \frac{s}{a} = \frac{t}{a + b} )</td>
</tr>
</tbody>
</table>
Three-Dimensional Figures

What’s Your View?

Lesson 33-1 Prisms and Pyramids

Learning Targets:
- Describe properties and cross sections of prisms and pyramids.
- Describe the relationship among the faces, edges, and vertices of a polyhedron.

SUGGESTED LEARNING STRATEGIES: Create Representations, Use Manipulatives, Self Revision/Peer Revision, Interactive Word Wall, Visualization

Len Oiler has gone into business for himself. He designs and builds tents for all occasions. Len usually builds a model of each tent before he creates the actual tent. As he builds the models, he thinks about two things: the frame and the fabric that covers the frame. The patterns for Len’s first two designs are shown by the nets below.

1. Build a frame for each model.

   Figure 1
   
   Figure 2

2. A **prism** is a three-dimensional solid with two congruent, polygonal, and parallel bases, whose other **faces** are called lateral faces. Knowing that prisms are named according to their base shape and whether they are right or oblique, refer to the first tent frame you constructed based on Figure 1.
   a. What is the shape of the two congruent and parallel bases?
   b. What is the shape of the three lateral faces?
   c. What is the best name for the solid formed by the net in Figure 1?
   d. Use geometric terms to compare and contrast right prisms and oblique prisms.

MATH TERMS

A **net** is a two-dimensional representation for a three-dimensional figure.

**Faces** are the polygons that make up a three-dimensional figure.

An **edge** is the intersection of any two faces.

A **vertex** is a point at the intersection of three or more faces.

In an **oblique prism**, the edges of the faces connecting the bases are not perpendicular to the bases. In a **right prism**, those edges are perpendicular to the bases.

DISCUSSION GROUP TIP

As you read and define new terms, discuss their meanings with other group members and make connections to prior learning.
3. A **pyramid** is a three-dimensional solid with one polygonal base, whose lateral faces are triangles with a common vertex. Knowing that pyramids are named according to their base shape and whether they are right or oblique, refer to the second tent frame you constructed based on Figure 2 on the previous page.

![Pyramid Diagram](image)

- **a.** What is the shape of the base?
- **b.** What is the best name for the solid formed by the net in Figure 2?
- **c.** Use geometric terms to compare and contrast right pyramids and oblique pyramids.

4. The figure shown is a hexagonal pyramid. Locate the net for this solid on the worksheets your teacher has given you.

![Hexagonal Pyramid](image)

- **a.** List the vertices of the pyramid.
- **b.** List the base edges.
- **c.** List the lateral edges.
- **d.** List the lateral faces.
Lesson 33-1
Prisms and Pyramids

5. Complete the table below. Nets for the pentagonal prism and pyramid can be found on the worksheets your teacher has given you. All listed figures are examples of polyhedra (singular: polyhedron).

<table>
<thead>
<tr>
<th>Figure Name</th>
<th>Number of Faces</th>
<th>Number of Vertices</th>
<th>Total Number of Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular prism</td>
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</tr>
<tr>
<td>Rectangular prism</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pentagonal prism</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Square pyramid</td>
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<td></td>
<td></td>
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<tr>
<td>Pentagonal pyramid</td>
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<th>7</th>
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<td>18</td>
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<td></td>
<td>4</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

6. List any patterns that you observe in the table above.

7. Make use of structure. Write a rule that expresses the relationship among \( F \), \( V \), and \( E \).

8. Three views of the hexagonal pyramid are shown below.
   a. Label each view as side, top, or bottom.

   ![Hexagonal Pyramid Views]

   b. Attend to precision. Explain the significance of the dashed segments in the views.

CONNECT TO HISTORY
Leonard Euler (1707–1783) was one of the first mathematicians to write about the relationship between the number of faces, vertices, and edges in polyhedrons. Leonard Euler was not just a “one-dimensional” man; he made contributions to the fields of physics and shipbuilding, too.
9. Sketch and label the bottom, top, and side views of this rectangular prism.

![Rectangular Prism Diagram]

10. If the bottom of the solid formed by the net shown is face B, sketch the top, front, and side views.

![Net Diagram with Points A, B, C, D, E]
Lesson 33-1
Prisms and Pyramids

11. Sketch the bottom, top, and side views of a square pyramid.

A **cross section** of a solid figure is the intersection of that figure and a plane.

12. **Make sense of problems.**
   a. Describe the two-dimensional shape of a cross section that is parallel to the base of a prism.

   b. Describe the two-dimensional shape of a cross section that is perpendicular to the base of a prism.

   c. Below are several cross sections of a hexagonal right pyramid. Label each cross section as being *parallel* or *perpendicular* to the base of the pyramid.
13. Sketch several cross sections that are parallel to the base of a square pyramid. Sketch several cross sections that are perpendicular to the base of a square pyramid. What patterns do you observe?

14. Use geometric terms to compare and contrast prisms and pyramids.

15. **Model with mathematics.** Consider the triangular prism below.

   a. Sketch several cross sections that are parallel to the bases of the triangular prism.
   b. Sketch several cross sections that are perpendicular to the bases of the prism.
   c. Describe the patterns you notice.
Lesson 33-1
Prisms and Pyramids

LESSON 33-1 PRACTICE

16. Consider the prism below.

![Prism Image]

a. Create a net for the figure.
b. Sketch and label the top, bottom, and side views of the figure.
c. Determine the shape of a cross section that is parallel to the bases.

17. Consider the polyhedron below.

![Polyhedron Image]

a. Identify the vertices of the figure.
b. Identify the shape of the two congruent and parallel bases.
c. Identify the shape of the lateral faces.
d. How many lateral edges compose the figure?
e. Name each lateral face.
f. List the base edges.
g. What is the best name for the polyhedron?

18. A polyhedron has 10 edges. Explain how to determine the sum of the number of faces and vertices that compose the figure. Then name the polyhedron.

19. Critique the reasoning of others. The design for a new toy is shown in the net below. Jamal classifies the toy as a hexagonal prism since there are six lateral faces that meet at one point. Do you agree with Jamal? Justify your response.
Learning Targets:
- Describe properties and cross sections of a cylinder.
- Describe properties and cross sections of a cone.

SUGGESTED LEARNING STRATEGIES: Group Presentation, Self Revision/Peer Revision, Visualization, Think-Pair-Share, Interactive Word Wall

A **cylinder** is the set of all points in space that are a given distance (radius) from a line known as the axis. Bases are formed by the intersection of two parallel planes with the cylinder. If the axis is perpendicular to each of the bases, the cylinder is called a right cylinder.

1. Label the axis, radius, and height in the diagram of the right cylinder.
2. Describe the shape and size of the cross sections that are parallel to the bases.

3. **Attend to precision.** Sketch an oblique cylinder.

4. Len is working on a pattern for a right cylinder. Which base is the correct size for this cylinder? Explain your answer.

5. Explain how the dimensions of the rectangle in the net in Item 4 relate to the cylinder once it is constructed.
Lesson 33-2
Cylinders and Cones

A cone is the union of all segments in space that join points in a circle to a point, called the vertex of the cone, that is not coplanar with the circle. The base of a cone is the intersection of a cone and a plane. If the segment that joins the center of the circle to the vertex is perpendicular to the plane that contains the center, the cone is a right cone and this segment is called the height of the cone. Otherwise, the cone is oblique.

6. Compare and contrast the shape and size of the cross sections of a right cone that is parallel to the base with the cross sections of a cylinder that is parallel to its base.

7. Reason abstractly. Describe the cross section of a right cone that is perpendicular to the base and contains the vertex of the cone.

When the tent business gets slow, Len makes lampshades, which are formed when a plane parallel to the base intersects the cone.

8. Len knows the height of the cone is 30 inches and the radius is 9 inches. He needs to know the area and circumference of the smaller circle, which is the top of the shade. (Assume the cross sections are 20 inches apart and parallel.)
   a. Label the center of the small circle P, and draw in \( PQ \). Identify two similar triangles in the figure. Explain why they are similar.
   b. Let \( r \) represent the radius of the small circle. Using corresponding sides of the similar triangles, write a proportion and solve for \( r \).
   c. Find the circumference and area of the smaller cross section.
Lesson 33-2
Cylinders and Cones

Check Your Understanding

9. Describe the shape of the cross sections that are perpendicular to the bases of a right cylinder.

10. A figure has one base. The distance from the center of the base to the vertex is greater than the height. What is the best name for the figure?

11. Emma and Malia are both studying cross sections of the right cylinder shown. Emma states that all planes that pass through the axis of the cylinder form cross sections in the shape of a circle. Malia says this is not always true. Provide a counterexample of Emma’s statement to prove that Malia is correct.

LESSON 33-2 PRACTICE

12. Make sense of problems. PVC is a type of plastic that is often used in construction. A PVC pipe with a 4 in. diameter is used to transfer cleaning solution into a process vessel. Explain how to compute the cross-sectional area of the pipe.

13. A graphic designer is working on a new label for a product that is packaged in a can. The radius of the cylinder is 12 cm. The distance between the top and bottom of the can is 15 cm.
   a. Describe the shape of the label prior to it being attached to the can.
   b. What are the dimensions of the label so that it completely covers the curved area of the can? Explain how you determined your answer.
   c. What is the area of the label, to the nearest square centimeter?

14. For each figure, determine the number of axes of symmetry and the number of planes of symmetry.
   a. \[ \text{right cylinder} \]
   b. \[ \text{right cone} \]
   c. \[ \text{oblique cylinder} \]
   d. \[ \text{oblique cone} \]
Lesson 33-3
Spheres and Solids of Rotation

Learning Targets:
- Describe properties and cross sections of a sphere.
- Identify three-dimensional objects generated by rotations of two-dimensional objects.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Create Representations, Visualization, Think-Pair-Share

Len provides spherical-shaped paper lanterns for his customers to use on camping trips. A sphere is created when a circle is rotated in space about one of its diameters. A sphere is the set of all points in space that are a given distance (radius) from a given point (center).

1. Consider the cross sections of a sphere.
   a. Describe the shape and size of the cross sections.
   b. Where will the largest of these cross sections be located?
   c. Where is the center of the largest possible cross section of a sphere located in relation to the center of the sphere?

2. The figure shown is a sphere.

   ![Diagram of a sphere with a chord and a diameter]

   a. Write a definition for a chord of a sphere. Draw and label a chord in the diagram.
   b. Write a definition for the diameter of a sphere. Draw and label a diameter in the diagram.
   c. Write a definition for a tangent (line or plane) to a sphere. Draw and label a tangent line and a tangent plane in the diagram.
   d. What must be true about a plane that is tangent to a sphere and a radius of the sphere drawn to the point of tangency? Explain your answer.
Lesson 33-3
Spheres and Solids of Rotation

A sphere is created by rotating a semicircle about a line, as shown in the figure below.

Rotating a plane about a given axis such that it forms a three-dimensional figure is known as a **solid of rotation**.

3. **Model with mathematics.** Sketch and name the solid generated by rotating rectangle $ABCD$ about line $m$. 

Right cylinders, cones, and spheres are often called **solids of rotation** because they can be formed by rotating a rectangle, triangle, or semicircle around an axis. In calculus, you will learn how to compute the volume of solids of rotation using a definite integral.

**CONNECT TO AP**

Right cylinders, cones, and spheres are often called **solids of rotation** because they can be formed by rotating a rectangle, triangle, or semicircle around an axis. In calculus, you will learn how to compute the volume of solids of rotation using a definite integral.
4. Sketch and name the solid generated by rotating the triangle about the $x$-axis.

5. Describe how Figure B was created from Figure A. Then name the various three-dimensional figures that make up Figure B.

6. **Use appropriate tools strategically.** Cut out any two-dimensional shape, tape it to a string or wooden stick, then hold the string/stick vertically and twirl it between your fingers to see the three-dimensional solid it forms.
   a. Sketch the three-dimensional figure formed by your model.
   b. Compare your model with a partner’s. Describe the differences in the figures formed. What accounts for these differences?
   c. Describe other tools you can use to create a solid of rotation model. Use your model to generate at least two more solid figures.
7. What is the ratio of the measure of the radius of a sphere to the measure of the radius of its great circle? Explain.

8. Two spheres intersect, as shown in the diagram below. Describe the intersection of the two spheres.

9. Can a solid of rotation about the $x$-axis generate the three-dimensional figure as shown? Explain.

**LESSON 33-3 PRACTICE**

10. How do the areas of the cross sections of a sphere compare to each other as you move farther away from the center of the sphere?

11. Identify by name the greatest chord in a sphere.

12. What figure is formed by rotating a square about line $m$?

13. Which of the following two-dimensional figures was rotated about the $y$-axis to form the three-dimensional figure shown?

14. **Model with mathematics.** Keiko designed a pendant for a necklace by rotating a square about a diagonal, as shown. What is the width of the pendant at its widest point?
ACTIVITY 33 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 33-1
1. A pyramid has 24 edges and 13 faces. Find the number of vertices in the figure.
2. A polyhedron has 7 faces and 15 edges. How many vertices does it have?
   A. 8  
   B. 10  
   C. 22  
   D. 24
3. Can you create a polyhedron with 17 vertices and 10 faces? Justify your reasoning.
4. An artist that specializes in metal work created the paperweight shown below.
   a. Name the shape of the base.
   b. Classify the faces in the paperweight.

5. Which of the following nets represents a triangular prism with isosceles triangle faces?
   A.  
   B.  
   C.  
   D.  

ACTIVITY 33 continued
Lesson 33-2

6. Which of the following figures can be formed with two identical circular discs and a rectangular piece of aluminum foil?
   A. square pyramid
   B. right cylinder
   C. oblique cone
   D. rectangular prism

7. A plane intersects a cone, as shown below.

   a. Describe the shape of the cross section formed.
   b. The base of the shape you identified in part a is the same measure as what part of the cone?
   c. What measure do the cone and the shape you identified in part a have in common?

8. Draw the possible cross sections of the cylinder shown.

9. A graphic designer is creating a label for a three-dimensional figure. The net of the figure is shown on the coordinate grid.

   What figure is represented by the net?
   A.   
   B.   
   C.   
   D.   

Lesson 33-3

10. If the lines of latitude on a globe represent small circles, what feature on a globe represents a great circle?

11. Are spheres a type of polyhedron? Why or why not?

12. What type of shape is formed when a semicircle is revolved about its diameter?
   A. a cone
   B. a circle
   C. a sphere
   D. a hemisphere

13. Name the three-dimensional figure formed when rectangle QRST is rotated about ST.

14. The axis of symmetry of a cone is the x-axis. Describe, using solids of rotation, how the cone was formed.

MATHEMATICAL PRACTICES
Reason Abstractly and Quantitatively

15. Three different cross sections of a cone are shown below. Are the cross sections in each figure the same shape? Explain.
Prisms and Cylinders

Exterior Experiences
Lesson 34-1 Surface Area of Prisms and Cylinders

Learning Targets:

• Solve problems by finding the lateral area or total surface area of a prism.
• Solve problems by finding the lateral area or total surface area of a cylinder.

SUGGESTED LEARNING STRATEGIES: Create Representations, Visualization, Think-Pair-Share

Brooke manages a group of tour guides and bush pilots for a company called Exterior Experiences. Brooke needs to order tents for a large group to be led into a wilderness area to view grizzly bears.

Mike is a bush pilot. He is planning to fly a group of Junior Rangers to a remote lake for a week of camping and trout fishing. They will be sleeping in tents that are in the shape of a triangular prism. The net below is a pattern for one of the Junior Ranger tents. Examine the two smaller right triangles at the bottom of the pattern. When they are zipped together, they form a triangle that is congruent to the equilateral triangle at the top of the pattern.

1. Find the area and perimeter of the equilateral triangular base.

2. To complete Item 1, you actually determined the missing zipper measure on the equilateral triangular base. Identify the theorem you used to determine the missing zipper measure.

3. Once the tent is constructed, what is the distance between the two bases?
Lesson 34-1
Surface Area of Prisms and Cylinders

4. Notice that the lateral faces of the prism form a larger rectangle.
   a. What are the dimensions of this larger rectangle?

   b. How do the dimensions of the larger rectangle relate to the triangular base and the distance between the two bases?

   c. The area of the largest rectangle is the sum of the areas of the lateral faces, known as the **lateral area**. Find the lateral area of this prism.

   d. If \( h \) represents the distance between the two bases in a prism, create a formula for finding the lateral area of a right prism in terms of \( h \) and the perimeter of the base, \( p \).

5. The **total surface area** of a prism is the sum of the lateral area and the area of the two bases.
   a. Calculate the total surface area for one of the Junior Ranger tents.

   b. Write a formula for calculating the total surface area, \( SA \), in terms of the lateral area, \( LA \), and the base area, \( B \).

6. **Reason abstractly.** In the formula you generated for the surface area in Item 5b, why is there a 2 as the coefficient in front of the base, \( B \)?

7. Mike sleeps in a tent with a different-shaped front and back than those of the other tents, as shown below. Find the lateral area and the total surface area of Mike's tent (including the floor). Show your work.

   ![Diagram of Mike's tent with dimensions provided]

MATH TERMS

The **lateral area** of a solid is the surface area of the solid, excluding the base(s).

The **faces** of a prism are its flat surfaces. The **bases** of a prism are the two congruent and parallel faces.

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Lesson 34-1
Surface Area of Prisms and Cylinders

Check Your Understanding

8. Identify the figure formed with two square bases and four rectangular lateral faces.
9. Consider the triangular prism below.

![Triangular Prism Diagram]

a. Identify the shapes and dimensions of the lateral faces.
b. Find the lateral area of the prism.
c. Identify the shapes and dimensions of each base.
d. What two measures can you now add together to find the total surface area of the prism? Compute the surface area of the prism.
e. If the dimensions of the base remain the same and the height is doubled to create a new prism, what will be the surface area of the new triangular prism?

Gino owns a company that manufactures cylindrical stuff sacks from water- and puncture-resistant fabric. Hikers and campers often use these stuff sacks to carry their tents and clothing when they go out on an excursion. The bases of the cylindrical sacks have a 6-inch radius, and the height is 15 inches.

10. Calculate the circumference and area of one of the bases of the stuff sack. Leave your answers in terms of π.
11. If the sack is laid out on a table, what shape is the lateral surface of the cylinder? What are its dimensions?

12. How do the dimensions of the lateral surface of the sack relate to the circular base and the height of the cylinder?

13. Determine the lateral area of a stuff sack.

14. Write a formula for calculating the lateral area, \( LA \), of a right cylinder in terms of the radius, \( r \), the height, \( h \), and \( \pi \).

15. Calculate the total surface area of one of Gino's stuff sacks.

16. Write a formula for calculating the total surface area, \( SA \), of a cylinder in terms of the lateral area, \( LA \), and the base area, \( B \).

Check Your Understanding

17. The pipe below is open on each end. Chester computes the surface area of the exterior surface of the pipe as \( 32\pi \) ft\(^2\). Maria computes the surface area as \( 24\pi \) ft\(^2\). Which student is correct? Justify your reasoning.

![Pipe Diagram]

18. A manufacturing company produces plastic cylindrical trash cans with heights of 5 feet and diameters of 2 feet. A 10 ft-by-12 ft sheet of plastic is fed into a cutting machine. If each machine cuts the three shapes needed to make a trash can, how much scrap plastic is left once the plastic leaves the machine? Show your work.
Lesson 34-1
Surface Area of Prisms and Cylinders

LESSON 34-1 PRACTICE

19. Find the lateral and total surface areas of a prism whose base is a right triangle with legs 5 in. and 12 in. and whose height is 3 in.

20. A birdhouse in the shape of a right prism has a lateral area of 90 ft². The base of the birdhouse is a regular pentagon with 4 ft sides. Find the height of the prism.

21. **Reason quantitatively.** A company that manufactures GPS devices ships the product to their customers using the shipping container shown. To cash in on the rebate, the customer needs to cut 2 inches off of the top of the container and mail it back to the manufacturer, along with the receipt for the device.
   a. What is the surface area of the original shipping container?
   b. Darcy states that to find the surface area of the shipping container after the rebate section is removed, you just need to find the sum of the areas of the four 2 in. sides and subtract it from the surface area of the original container. Judd says that Darcy is forgetting to do something. Identify the step that Darcy needs to include. Explain.

22. In a convective type of heat exchanger, a smaller cylinder is placed inside a larger cylinder, as shown.
   a. Write the ratio of the lateral area of the smaller cylinder to the lateral area of the larger cylinder.
   b. Write the ratio of the surface area of the smaller cylinder to the surface area of the larger cylinder.

23. A cylinder has radius 4 inches and height 5 inches. If the radius of the cylinder is tripled and the height remains the same to create a new cylinder, what will be the difference in the surface areas of the two cylinders? Give your answer in terms of π.
Learning Targets:
- Solve problems by finding the volume of a prism.
- Solve problems by finding the volume of a cylinder.

**SUGGESTED LEARNING STRATEGIES:** Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Interactive Word Wall

Extreme Exteriors gives their tour guides a thermos before every excursion. Depending on the excursion, the thermos is either a right or an oblique cylinder. Sometimes the thermos is easier to carry on a bike or in a kayak if it is oblique.

Inside each thermos is a small filter in the shape of a hexagon, which uses thin, hexagonal-shaped filter paper. Extreme Exteriors always keeps a container of filter papers on hand.

Examine the filter paper container shown. The base of the container is congruent to each filter paper.

1. Name the figure that describes the container.

2. If the area of each filter paper is \( B \), write a formula that finds the volume of filter paper the container holds.

3. Suppose the radius of the container is 2 in. and its height is 4 in.
   a. What is the volume of filter papers that fits in the container? Explain how you arrived at your answer.
   b. A new container is created in which the radius is tripled and the height remains the same. Describe how the change in dimension affects the volume of the papers that fits the container. Justify your answer.

Two cylindrical-shaped thermoses have radius \( r \) and height \( h \).
Lesson 34-2
Volume of Prisms and Cylinders

4. Suppose a plane parallel to the base of the cylinder passes through the cylinder.
   a. Describe the cross section of each cylinder.
   
   b. What is the area of the cross section of each cylinder?

5. Recall the volume formula for a prism in Item 2, where \( B \) is the base area and \( h \) is the height. What is the base area of the cylinder? How does this compare to your answer in Item 4b?

6. Suppose each cylinder was dissected into wedges and rearranged to form a solid, like that of a prism. Use what you know about the volume formula of a prism to determine the volume formula of a cylinder.

Example A
The volume of the cylinder shown is 450 ft\(^3\) and the diameter is 8 ft. What is the height of the cylinder?
Use the volume formula for a cylinder, \( V = Bh \), or \( \pi r^2 h \), and substitute known values.

\[
\frac{V}{\pi r^2} = \frac{\pi r^2 h}{\pi r^2} \quad \text{Divide to isolate } h.
\]

\[
\frac{V}{\pi r^2} = h
\]

\[
\frac{450}{\pi (4)^2} = h
\]

\[
8.95 \approx h
\]

The height of the cylinder is about 8.95 ft.

Try These A
a. The volume of a cylinder is 220 cm\(^3\), and the radius is 5 cm. What is the height of the cylinder, to the nearest tenth?

b. The volume of a cylinder is 90 m\(^3\), and the circumference of one of the bases is 27 m. Determine the height of the cylinder.

c. The volume of a rectangular prism is 120 ft. The width is 15 ft, and the height is 4 ft. What is the length of the prism?
d. The volume of a rectangular prism is 480 cubic feet.
i. What are possible dimensions of the prism?

ii. If the dimensions of the base are doubled and the height remains the same to create a new prism, what will be the volume of the new rectangular prism in cubic feet?

iii. Describe how the change in dimensions affects the volume of the prism

Brooke is looking to purchase one of two new styles of tents. One tent is in the shape of a half-cylinder, and one is in the shape of a triangular prism. Since the price of the tents is dependent on the area of material, Brooke wants to find the tent with the greatest volume (living area) using the least amount of material.

Now, work through a multipart problem that uses both surface area formulas and volume formulas.

7. First consider the volume of each tent.

a. Write the formula for the volume of a cylinder. How can you adapt the formula to compute the volume of Tent A?

b. Find the volume of Tent A. Show your calculations.

c. Name the figure that describes the base of Tent B. What is the area of each base?

d. Find the volume of Tent B. Show your calculations.

e. Which tent has the greater volume?
8. Next, consider the surface area of each tent. Keep in mind that each tent has a floor.
   a. Name the shapes that describe the base and the lateral faces of Tent A. How can you adapt the surface area formula of a cylinder to compute the surface area of Tent A?

   b. Find the surface area of Tent A. Show your calculations.

   c. Name the shapes that describe the base and the faces of Tent B. Find the dimensions of each face and base.

   d. Find the surface area of Tent B. Show your calculations.

   e. Which tent has the greater surface area?

9. Make sense of problems. Complete the table for the volume and surface area of each tent. Which tent has the greatest volume with the least amount of material? Explain how you determined your answer.

<table>
<thead>
<tr>
<th></th>
<th>Tent A</th>
<th>Tent B</th>
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<tbody>
<tr>
<td>Volume</td>
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</tr>
<tr>
<td>Surface Area</td>
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</tbody>
</table>
Check Your Understanding

10. **Attend to precision.** In Items 8 and 9, why are the units for perimeter, surface area, and volume different?

11. True or false? If a right prism and an oblique prism have congruent bases, then the volume of the right prism can never be equal to the volume of the oblique prism. If true, state the theorem that supports this fact. If false, provide proof.

12. If a cylindrical container does not have a lid, does it have an effect on the measure of the volume, the surface area, neither, or both? Justify your answer.

**LESSON 34-2 PRACTICE**

13. The length of the cylindrical tube shown is 10 ft. What is the volume of the tube given that the circumference of one of the bases is $48\pi$ ft?

14. **Model with mathematics.** A United States gold dollar has a radius of 26.5 mm and a thickness of 2 mm. Calculate the total volume of gold in a stack of 25 gold dollars, to the nearest mm$^3$.

15. The bases of a triangular prism are equilateral triangles with sides of 8 ft. The height of the prism is 5 ft.
   a. Determine the volume of the prism, to the nearest cubic foot.
   b. The height of the prism is doubled and the dimensions of the base remain the same to create a new prism. Describe how the change in dimension affects the volume of the prism.
   c. The height of the prism remains the same and the dimensions of the base are tripled to create a new prism. Describe how the change in dimension affects the volume of the prism.

16. The volume of a rectangular prism is 1200 ft$^3$. The bases are squares, as shown. What is the length of the prism?

17. If two of the same type of three-dimensional figure have the same volume, is this a guarantee that the two figures have the same surface area? Explain.
ACTIVITY 34 PRACTICE
Write your answers on notebook paper.
Show your work.

Lesson 34-1

1. A cell phone is packaged in a rectangular cardboard container prior to shipment. The height of the container is 12 in., and the base has dimensions of 5 in. by 7 in.
   a. Compute the lateral area of the cardboard container.
   b. Compute the surface area of the cardboard container.
   c. The fee to ship the phone to a customer is $4.50 plus $0.05 per square inch of the exterior surface of the container. Compute the total shipping fee.

2. A stepping-stone has the shape of a right prism. It has a height of 10 ft and a base in the shape of a rhombus. The diagonals of the rhombus measure 12 ft and $12\sqrt{3}$ ft.
   Which of the following measures is the lateral area of the stepping-stone?
   A. 120 ft²
   B. 480 ft²
   C. 560 ft²
   D. 960 ft²

3. The lateral surface of the cylindrical can shown below is to be covered with a label. Determine the greatest area of label that can be used to cover the lateral surface of the can, without overlap.

4. Consider the right triangular prism below.

   a. Compute the lateral area of the prism.
   b. Compute the surface area of the prism.

5. A cereal company is designing a box for a new product. A right rectangular prism-shaped box stacks best on store shelves. The cereal will all fit into a prism that measures 10 in. long, 2.5 in. wide, and 14 in. high. The art department is designing graphics for the package and needs to know how much room they have to work with.
   What is the total surface area, in square inches, of the cereal box?
   A. 56 in.²
   B. 105 in.²
   C. 350 in.²
   D. 400 in.²
6. A pipe connects a soybean oil storage tank and a process vessel. Determine the lateral area of the pipe given that it is 12 ft long and 4 in. in diameter.

7. Determine the height of a right prism with lateral area 45 sq ft if the base is a regular hexagon each of whose sides is 5 ft.

8. What is the approximate surface area of the figure formed by the net shown?

9. Determine the volume of the right triangular prism.

10. What is the radius of this cylinder if its volume is 2262 m$^3$?

11. A cylindrical tank with a diameter of 8 feet and a height of 14 feet is being filled with water. If the water is entering the tank at a rate of 2 cubic feet per second, how long will it take for the tank to be half full?

12. A wooden storage trunk has a height of 4 ft, a length of 6 ft, and a width of 2 ft. The trunk is to be coated with two coats of polyurethane. What measure is needed to determine the total amount of polyurethane needed?

   A. volume
   B. weight
   C. lateral area
   D. surface area

13. The diameter of an oblique cylinder is 32 cm and the height is 4 cm. What is the volume of a right cylinder with the same dimensions?

14. The lateral areas, $LA$, of a cube and a right cylinder are equal. The figures even have the same height, $h$.
   a. Write expressions for the lateral area of each figure.
   b. Write an expression that can be used to determine the radius of the cylinder. Show how you determined your answer.
Learning Targets:
• Solve problems by finding the lateral area or total surface area of a pyramid.
• Solve problems by finding the lateral area or total surface area of a cone.

SUGGESTED LEARNING STRATEGIES: Create Representations, Visualization, Think-Pair-Share

The way a product is packaged is an important marketing element for manufacturers, and in today’s world, it is an important environmental concern for consumers.

Luis is leading a committee to change some of the current packaging materials used by his company that have been deemed nonrecyclable to more environmentally friendly materials. Many of the environmentally friendly materials, however, do not have the same strength or durability as the packaging materials currently in use. The committee realizes that if the materials are changed, then the shape of the actual package will likely have to change to ensure strength and durability.

The committee consists of representatives from the marketing department and the advertising department, a product engineer, and a packaging engineer. To begin, the marketing representative provides the committee with feedback that consumers are attracted to products packaged in pyramid- and cone-shaped containers.

Surface area is a critical concern for the advertising representative, as it dictates how much space is available for print. Before the committee goes any further, the advertiser wants to determine the surface area of a cone and pyramid.

1. Consider the tetrahedron, a special type of pyramid with an equilateral triangle for a base and equilateral triangles for each of its three sides. Suppose the length of each edge of the container is 8 ft.

   a. Find the area of each face.

   b. Find the total surface area of the four faces.

MATH TIP
The formula for finding the area of an equilateral triangle, given the length of a side $s$, is $A = \frac{s^2 \sqrt{3}}{4}$.
c. Use the formula for calculating the area of an equilateral triangle to derive a formula for the total surface area of one of these pyramids in terms of the length of an edge, \( s \).

d. The packaging engineer provides a 27.75 sq. yd sample of environmentally friendly material to create a model of this tetrahedron-shaped container. Find the dimensions of the pyramid.

2. Consider the square pyramid. Suppose each side of the square base of the pyramid is 12 ft, and the length of each lateral edge is 10 ft.

   ![Diagram of a square pyramid]

   a. Determine the perimeter and the area of the base.

   b. How can you determine the height of a triangular face?

   c. The height of a triangular face is the slant height. Draw one of the slant heights, and determine its length. Then calculate the area of one of the triangular faces.

   d. The lateral area of a pyramid is the sum of the areas of the lateral faces. Calculate the lateral area of the pyramid.

MATH TERMS

The slant height of a regular pyramid, such as a square pyramid, is the distance between the vertex and the base edge of the pyramid. It is the altitude of a lateral face.
Lesson 35-1
Surface Area of Pyramids and Cones

e. If $l$ represents the slant height of the pyramid, create a formula for finding the lateral area, $LA$, of a right pyramid in terms of $l$ and the perimeter of the base, $p$.

f. The total surface area of a pyramid is the sum of the lateral area and the area of the base. Calculate the total surface area for the square pyramid.

g. Write a formula for calculating the total surface area, $SA$, of a right pyramid in terms of the lateral area, $LA$, and the base area, $B$.

3. Use your formulas in Items 2e and 2g to find the lateral area and total surface area for a regular tetrahedron, each of whose edges is 8 feet in length.

Check Your Understanding

4. Name the shapes of the lateral faces of a regular pyramid.

5. Can a pyramid ever have quadrilaterals for lateral faces? Explain.

6. Each side of the base of a square pyramid is 5 ft. The slant height of the pyramid is 4 ft.
   a. Calculate the lateral area of the pyramid.
   b. What is the surface area of the pyramid?
   c. The slant height of the pyramid is doubled and the dimensions of the base remain the same to create a new pyramid. Describe how the change in dimension affects the lateral area of the pyramid.

7. Each edge of a regular tetrahedron is 12 in.
   a. Calculate its lateral area.
   b. What is its surface area?
Lesson 35-1
Surface Area of Pyramids and Cones

The advertising representative would like to have a formula for calculating the surface area for a cone-shaped package, in terms of the dimensions. She knows the formula for the area of the base of a cone, which is a circle. It’s the lateral area that is troubling her. The packaging engineer guides her through an explanation by displaying several patterns for the curved surface of a cone, the lateral surface.

8. Use appropriate tools strategically. The engineer points out that each pattern is actually a sector of a circle. Find the patterns for the curved surface of several cones on the worksheet your teacher provided. Cut out each pattern and build the lateral surface for each cone. Trace each circular base on a sheet of paper and answer the questions below.

a. As the lateral area of the cone decreases, what happens to the height of the cone?

b. As the lateral area of the cone decreases, what happens to the radius of the base?

c. As the lateral area of the cone decreases, what happens to the slant height of the cone?

d. Which part of the cone corresponds to the radius of the sector?
Lesson 35-1
Surface Area of Pyramids and Cones

The engineer starts each pattern for the lateral surface with a large circle with radius $l$. Then he cuts out a sector, as shown below.

9. Make sense of problems. Use the figures above to respond to each of the following.
   a. Write an expression for the area and circumference of the circle in terms of $l$ and $\pi$.
   
   b. Write the ratio of the area of the sector $A_S$ to the area of the circle.
   
   c. Write the ratio of the length of the arc of the sector $a$ to the circumference of the circle.
   
   d. Explain why the two ratios in parts b and c can be used to create a proportion.
   
   e. Write the proportion from part d, and solve this proportion for $A_S$. Simplify your answer.
   
   f. How does $a$, the length of the arc of the sector, relate to the base of the cone?
   
   g. Write an expression for the lateral area of a cone, $LA$, in terms of $l$, the radius of the base, $r$, and $\pi$.
   
   h. Write a formula for calculating the total surface area, $SA$, in terms of the lateral area, $LA$, and the base area, $B$. 
Lesson 35-1
Surface Area of Pyramids and Cones

Check Your Understanding

10. Compare the formulas for surface area of cones and pyramids. Explain why the base area of a cone is not computed using the same formula as the base area for a pyramid.

11. The diameter of the base of a cone is 10 centimeters. The slant height of the cone is 7 centimeters.
   a. Determine the lateral area of the cone.
   b. What is the surface area of the cone?

12. A plastic cone-shaped package is 10 inches in diameter and 12 inches high.
   a. Make a sketch of the pyramid and label the dimensions.
   b. Find the lateral area.
   c. Compute the total surface area of the package.

13. Describe a step-by-step method you can use to find the height of a cone if you know the lateral area and the base area. Then use your method to find the height of a right cone whose lateral area is $136\pi \text{ ft}^2$ and base area is $64\pi \text{ ft}^2$.

LESSON 35-1 PRACTICE

14. Calculate the lateral area and surface area of a pyramid whose height is 8 ft and whose base is a square with 12 ft sides.

15. Find the lateral area of a right pyramid whose slant height is 18 mm and whose base is a square with area 121 mm$^2$.

16. Reason quantitatively. Calculate the surface area of the cylindrical rocket and “nose cone” if the slant height of the nose cone is 8 ft.

17. Find the lateral area and surface area for a right cone whose slant height forms a 45° angle with the base and whose radius is 8 mm.

18. The product engineer shows the committee a product he is working on that is cylindrical in shape with a conical top. Find the total surface area of the product given the diameter is 24 ft, the height of the cylindrical portion is 6 ft, and the height of the conical portion is 5 ft.
Learning Targets:

- Solve problems by finding the volume of a pyramid.
- Solve problems by finding the volume of a cone.

SUGGESTED LEARNING STRATEGIES: Visualization, Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Interactive Word Wall

For any new package design, the packaging engineer needs to confirm it can meet the volume requirements for the product. Luis asks the packaging engineer to explain how volume is computed for packages shaped like a pyramid or cone.

Consider the three potential package designs below. Each pyramid has the same base area and the same height.

1. **Model with mathematics**. The packaging engineer manipulates the pyramid into the figure shown. What figure do the three pyramids form?
2. A cube is a type of prism.
   a. What is the formula for volume of a prism?
   
   b. How many pyramids form the cube?
   
   c. Do the pyramids within the cube have the same volume? How do you know?
   
   d. Describe how the formula from Item 2a can be adapted to create a formula for the volume of one of the square-based pyramids.
   
   e. Write the formula for the volume of one of the square-based pyramids.

3. Describe how **Cavalieri’s Principle** can be used to justify the formula in Item 2e for any square-based pyramid. Remember to use complete sentences and words such as and, or, since, for example, therefore, because of, by the, to make connections between your thoughts.

4. **Critique the reasoning of others.** Keith states that the formula in Item 2e does not apply to a pyramid with any polygonal base. Angela states that the formula does apply, and that only the formula used to compute $B$ will be different. With whom do you agree and why?
Example A
Find the volume of a package shaped like a square pyramid, with the given measures.

Step 1: Find the length of the base.
The distance between the height of the pyramid and the slant height is one-half the base. Use the Pythagorean Theorem to find this distance.

\[ d^2 = 34^2 - 30^2 \]
\[ d = 16 \]
The length of the base is 2(16) = 32 in.

Step 2: Use the volume formula for a pyramid, \( V = \frac{1}{3} Bh \), and substitute known values.

\[ V = \frac{1}{3} (32)(32)(30) \]
\[ V = 10,240 \text{ in.}^3 \]

Solution: The volume of the pyramid is 10,240 in.³

Try These A

a. The height of a pyramid is 24 ft. The area of the base is 22 ft². Compute the volume of the pyramid.

b. The volume of a pyramid is 220 cm³ and the base area is 10 cm². What is the height of the pyramid?

c. The slant height of a square pyramid is 10 feet. One side of the base of the pyramid is 12 feet. What is the volume of the pyramid?
Now, consider a package design in the shape of a cone. The packaging engineer develops the volume formula by inscribing polygonal pyramids of varying bases in a cone.

5. Write the formula for the volume of a pyramid.

6. Examine the base of each cone. As the number of sides increases in the polygonal pyramid, is the space between the pyramid base and the circular base of the cone increasing or decreasing?

7. Reason abstractly. Suppose the number of sides of the polygonal pyramid inscribed in the cone continued to increase. What shape would the base of the pyramid resemble?

8. Write the area formula for a circle.

9. Use the formulas in Items 5 and 8 to write the formula for volume of a cone.
Lesson 35-2
Volume of Pyramids and Cones

10. Compute the volume of the cone with a radius of 6 cm and a height of 15 cm, to the nearest tenth.

11. A container of nuts in the shape of a right cone has a diameter of 18 cm. The height of the container is 7 cm. What is the volume of nuts that fits in the container?

12. Compare and contrast the volume formulas for cones and pyramids.

13. In terms of units of measure, how does the surface area of a cone differ from the volume of a cone?

14. A perfume bottle in the shape of a right cone has a diameter of 6 in. and a slant height of 5 in. How much perfume can the bottle hold, to the nearest cubic inch?

Check Your Understanding

15. The height of a cone is 10 ft. The area of the base is 22 ft². Compute the volume of the cone.

16. The volume of a cone is 80 cm³ and the base area is 5 cm². What is the height of the cone?

17. The volume of a right cone is 400 ft³ and the height is 25 ft. Find the slant height of the cone, to the nearest tenth.

18. The base of a right pyramid has dimensions of 5 m by 6 m. The height of the pyramid is 12 m. What is the volume of the pyramid?

19. Make sense of problems. Peanuts are sold at the baseball game for $3.75. The peanuts are packaged in boxes shaped as rectangular prisms. The height of the box is 7 inches and the base of the box has an area of 3 in². For the tournament games, management decided to package the peanuts in a pyramid design rather than as a prism. The pyramid has the same height and base area as the prism.
   a. Compare the volumes of the prism and the pyramid.
   b. Determine a price for the peanuts in the pyramid design that is proportionally fair to the original price. Explain your answer.
Learning Targets:
• Apply concepts of density in modeling situations.
• Apply surface area and volume to solve design problems.

SUGGESTED LEARNING STRATEGIES: Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Interactive Word Wall

Luis has learned that before they make any decisions about changing packaging materials, he needs to make sure he is within budget for the cost of the packaging materials and the cost of shipping the product.

The new material under consideration for packaging food products costs $0.45 per square foot. The shipping cost is $1.90 per pound. The table below shows the density for four different food products packaged in bulk and shipped from Luis’s company.

<table>
<thead>
<tr>
<th>Material</th>
<th>Density (lb/ft³)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shelled almonds</td>
<td>30</td>
</tr>
<tr>
<td>Apple seeds</td>
<td>32</td>
</tr>
<tr>
<td>Coffee beans</td>
<td>22</td>
</tr>
<tr>
<td>Soybean hulls</td>
<td>6</td>
</tr>
</tbody>
</table>

Luis is analyzing the cost of the following two package designs to determine which package makes the most sense to use for coffee beans.

1. Model with mathematics. Consider the cone. The diameter is 5 ft and the slant height is 6 ft.
   a. Determine the surface area of the cone. Then find the cost of the packaging material.
   b. Determine the volume of the cone.
   c. What is the relationship among volume, density, and mass?
   d. Find the mass of the cone filled with coffee beans.
   e. Find the shipping cost of the cone.
   f. What is the total cost for packaging and shipping the cone-shaped package filled with coffee beans?
2. Make a prediction as to whether the total cost for packaging and shipping the pyramid-shaped package will be greater than, less than, or the same as the cone-shaped package. Explain your reasoning.

3. Determine the validity of your prediction by analyzing the pyramid package.
   a. Find the cost of the packaging material for the pyramid design.
   b. Find the shipping cost of the pyramid.
   c. What is the total cost for packaging and shipping the cone-shaped package?

4. Compare the ratio of total cost to ship each package to volume shipped. How did your prediction in Item 2 compare to your findings?

5. Which food product in the table would be the least expensive to ship? Explain how you determined your answer.
Lesson 35-3

Density

6. Apple seeds are packaged in a container shaped like a square pyramid. Coffee beans are packaged in an identical container. Do the filled packages have the same mass? If not, explain how you can determine the product with the greater mass.

7. Shelled almonds are packaged in a cylindrical container with a 0.5-foot diameter and a height of 0.6 ft. What is the approximate mass of the almonds that fill the container, to the nearest tenth of a pound?

LESSON 35-3 PRACTICE

8. Butter has a density of 54 lb/ft³. It is packaged into a 5 in.-by-1.27 in.-by-1.27 in. cardboard prism. One package of butter contains four identical prisms. What is the approximate mass of one package of butter?

9. Two containers, a cone and a cylinder, both have a height of 6 ft. The bases have the same diameter of 4 ft. Both containers are filled with shredded paper that has a density of 12 lb/ft³.
   a. How do you know which container has the greatest mass?
   b. Make a generalized statement about the difference in volume of a cone and a cylinder given the same diameter and height.
   c. Determine the mass of each container. Are your answers reasonable? Explain how you know.

10. Construct viable arguments. A company needs to package crushed sandstone to send to a customer. Sandstone has a density of 1281 kg/m³. The customer asked if the company could send the sandstone in a Styrofoam container. Styrofoam has a density of 70 kg/m³. Should the company use a Styrofoam container to send the sandstone? Explain.
ACTIVITY 35 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 35-1
1. The figure below is a square pyramid with height 10 cm and base edge 20 cm. Calculate each of the following:
   
   A. the slant height
   B. the length of a lateral edge
   C. the length of the apothem of the base
   D. the length of the radius of the base

2. Paper cups used in some water dispensers are conically shaped. The cups have a diameter of 6 cm and a height of 8 cm. About how much paper is needed to make one of these cups, assuming there is negligible overlap?
   
   A. 24 cm²
   B. 48 cm²
   C. 109 cm²
   D. 151 cm²

3. Calculate the lateral and surface areas of a cone whose height is 12 in. and whose slant height is 15 in.

4. Consider the cone below.

   a. Compute the lateral area of the cone.
   b. Compute the surface area of the cone.

5. What is the slant height of a right pyramid whose lateral area is 90 ft² and whose base is a regular hexagon with a 3 ft side?
   
   A. 3 ft
   B. 10 ft
   C. 30 ft
   D. 270 ft

Lesson 35-2
6. Compare and contrast the pyramid and prism formulas.

7. A square pyramid has a volume of 196 cm³. Determine the height of the pyramid if the edges of the base measure 7 cm.

8. A cylinder has a volume of 200 in.³ Determine the volume of a cone whose radius and height are equal to that of the cylinder.
9. Determine the volume of the figure shown.

A. 114 cm³  
B. 126 cm³  
C. 340 cm³  
D. 648 cm³

10. The volume of a square pyramid is $66 \frac{2}{3}$ ft³. The height is 8 feet. What are the dimensions of the base?

Lesson 35-3

11. The density of steel is approximately 0.28 lb/in.³
Find the mass of a solid cube of steel with 24 in. sides.

12. A cargo container in the shape of a rectangular prism has a height of 4 ft, a length of 6 ft, and a width of 3 ft. The container is filled to capacity with a solid material. The total mass of the container is 1080 lb. Which of the following is the most likely material in the container?
A. cat food: density 20 lb/ft³  
B. glass beads: density 120 lb/ft³  
C. golf tees: density 15 lb/ft³  
D. lima beans: density 45 lb/ft³

13. The diameter of an oblique cylinder is 0.16 m and the height is 0.5 m. What is the approximate mass of a right cylinder with the same dimensions if it is filled with cottonseed? The density of cottonseed is 352 kg/m³.

14. A large process vessel, in the shape of a cone, is filled with maple syrup. The density of maple syrup is 1362 kg/m³. The height of the vessel is 4 m and the diameter is 1 m. What is the mass of the maple syrup in the vessel, to the nearest tenth?

MATHEMATICAL PRACTICES

Make Sense of Problems and Persevere in Solving Them

15. The figure below is a regular hexagonal right prism with a regular hexagonal pyramid carved into a base of the prism. Find the total surface area of the figure if the height of the prism and pyramid is 8 cm, and the length of one side of the hexagonal base of the prism and pyramid is 6 cm.
Play-Tel Toys has entered talks to create action figures for the latest Hollywood blockbuster film. The film’s main character, a robot named Arigato, is the focus of the marketing campaign.

In order to do a cost analysis for creating the action figures, Play-Tel designers must determine the surface area and volume of the robot’s body parts. Although the final shapes will be crafted in much greater detail, each body part will first be molded into the solids, as indicated in the diagram.

1. Complete the table based on the dimensions provided in the diagram.

<table>
<thead>
<tr>
<th>Surface Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylindrical Head</td>
<td></td>
</tr>
<tr>
<td>Conical Hat</td>
<td></td>
</tr>
<tr>
<td>Right Cylinder Torso</td>
<td></td>
</tr>
<tr>
<td>Right Cylinder Arm</td>
<td></td>
</tr>
<tr>
<td>Square Prism Leg</td>
<td></td>
</tr>
</tbody>
</table>

2. Each piece of the action figure will be painted before assembly. If it costs $0.02 per square centimeter to paint, determine the cost to paint each action figure.

3. The designers also need to determine the total amount of plastic to be used in the molds for each robot. If each cubic centimeter of plastic costs $0.03, determine the total cost to mold each action figure.
4. The designers are discussing the type of plastic material to use for the robot. The specified mass for the action figure is 1.34 kg. Use the density table to determine the type of plastic that would meet the mass specification. Justify your answer.

<table>
<thead>
<tr>
<th>Plastic Material</th>
<th>Density g/cm³</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polyethylene</td>
<td>0.96</td>
</tr>
<tr>
<td>Polypropylene</td>
<td>0.93</td>
</tr>
<tr>
<td>Poly Vinyl Chloride</td>
<td>1.47</td>
</tr>
</tbody>
</table>

Learning Targets:
• Solve problems using properties of spheres.
• Solve problems by finding the surface area of a sphere.

SUGGESTED LEARNING STRATEGIES: Create Representations, Visualization, Interactive Word Wall

“Cavalieri’s Kitchen Wows Customers with Volumes of Food and Fun!” was the headline in the latest edition of Food Critics magazine. When asked the secret of the restaurant’s success, head chef and owner Greta stated, “I combine my math background with my love of cooking to create interesting, appealing, and delicious food.”

One of Greta’s specialties is her ability to work with spherically shaped designs. A sphere is a three-dimensional shape such that all points are an equal distance from a fixed point. The fixed point is the center of the sphere and the distance from the fixed point is the radius of the sphere.

Example A

Greta is preparing a cake in the shape of a hemisphere, as shown below. She plans to wrap a piece of licorice around the cake, 4 inches up from the base. Given that the radius of the hemisphere is 6 inches, what is the total length of licorice needed?

Step 1: Use the Pythagorean Theorem to find the radius of the ring, y.

\[ 4^2 + y^2 = 6^2 \]
\[ y^2 = 20 \]
\[ y = \sqrt{20} \]

Step 2: To find the distance around the cake, use the circumference formula for a circle.

\[ C = 2\pi r \]
\[ C = 2\pi \sqrt{20} \]
\[ C \approx 28.1 \text{ in.} \]

Solution: The length of licorice needed is 28.1 in.
Lesson 36-1
Surface Area of Spheres

Try These A

a. The diameter of a sphere is 40 ft. A plane, parallel to the center of the sphere, passes through the sphere at a distance of 12 ft from the center. Find the radius of the circular intersection.

![Diagram of a sphere with a plane cutting through it at a distance of 12 ft from the center]

b. Find the circumference and area of the circular intersection of the plane in Part a.

Cavalieri’s Kitchen is a popular restaurant with fishermen. Every day Greta’s staff prepares lunches for fishermen to take on their charter fishing trips down the nearby river. Not only is the food delicious, but the fishermen appreciate that the lunches are packed by Cavalieri’s Kitchen in waterproof, spherical-shaped containers. The fishermen tie the containers to their boats and allow them to float along in the water. The food and drinks inside the containers stay cold because the river’s primary sources are spring water and melting snow.

The formula for calculating the surface area of a sphere is $A = 4\pi r^2$, where $r$ represents the radius of the sphere.

1. Calculate the surface area for one of the spherical containers if the radius is 8 inches.

2. Attend to precision. Determine the radius of a sphere whose surface area is 1256.64 cm². Round your answer to the nearest centimeter.
Lesson 36-1
Surface Area of Spheres

Each evening, the fishermen return the empty containers to Cavalieri’s Kitchen and typically stay for dinner. The restaurant is an interesting structure, as it is a hemisphere on a wooden platform.

3. Make use of structure. Describe where a plane would have to intersect a sphere in order to form two hemispheres. Explain your answer.

4. Write a formula for calculating the surface area of the curved surface of the restaurant in terms of its radius, \( r \).

5. Calculate the surface area of the curved surface of the hemisphere with radius 10 feet. Do not include the area of the floor.

6. Use the formula from Item 4 to verify the formula for the surface area of a sphere in terms of its radius, \( r \).

MATH TERMS
When a plane passes through the center of a sphere, the intersection is known as a great circle.

CONNECT TO AP
In calculus, you will learn how to derive the formula used to calculate the surface area of a sphere using limits or a definite integral.
Lesson 36-1
Surface Area of Spheres

Check Your Understanding

7. Name the shape of the intersection when a plane intersects a sphere in more than one point.
8. A sphere can be divided into four equal quadrants. What is the surface area of each quadrant? Explain how you determined your answer.
9. Name the figure that forms the base of a hemisphere.
10. Write a formula for computing the total surface area of a hemisphere, including the base.
11. Calculate the surface area of a sphere whose radius is 6 cm.
12. A tent is made in the shape of a hemisphere with an 18 ft diameter. How much fabric is used to make the tent, including the floor?

LESSON 36-1 PRACTICE

13. Find the surface area of a sphere, in terms of \( \pi \), given a radius of 15 in.
14. The surface area of a sphere is \( 144\pi \text{ ft}^2 \). What is the radius?
15. A plane, parallel to the center of the sphere, passes through the sphere at a distance of 12 cm from the center. The radius of the circular intersection is 9 cm.
   a. Find the diameter of the sphere.
   b. What is the area of the great circle?

16. A sphere is inscribed in a cube. The surface area of the sphere is \( 196\pi \text{ in}^2 \). What is the measure of the edge of the cube?

17. Make sense of problems. Cavalieri’s Kitchen makes fresh orange juice each morning. The 64 oranges that Greta’s staff peels each morning are spherical in shape. The radius of each orange is approximately 6 cm. The orange peels are discarded in a compost pile. How many square centimeters of orange peel are discarded each morning?
Learning Targets:
- Develop the formula for the volume of a sphere.
- Solve problems by finding the volume of a sphere.

**SUGGESTED LEARNING STRATEGIES:** Think-Pair-Share, Create Representations, Visualization

Greta is promoting a new menu item called the Blueberry Bubble. She tells her morning customers, “The hemisphere of blueberry pancakes that I am creating will absolutely melt in your mouth!” One of the fishermen, a regular morning customer, asks Greta to explain how the pancakes will follow the same mathematical principles as her other famous menu items. She states that, like a cone, a hemisphere is simply a fractional portion of a cylinder.

Greta uses three different stacks of vanilla pancakes to show her customers, placing one blueberry pancake in each stack, as shown below.

1. Describe the shape of each blueberry pancake.

2. Write the volume formula for the shape identified in Item 1.

3. How far is each pancake above the center of the corresponding figure?

4. Write expressions for the cylinder stack.
   a. Write an expression for the volume of the single blueberry pancake in the cylinder stack.

   b. **Reason abstractly.** Write an expression for the number of blueberry pancakes that will fill the entire cylinder.

   c. Use parts a and b to write an expression for the total volume of blueberry pancakes that will fill the cylinder stack.
5. Notice that the radius of the blueberry pancake in the double cone is \( x \). Write an expression for the volume of the single blueberry pancake in the double cone stack.

6. The volume of the single blueberry pancake in the sphere is equal to the volume of the one in the cylinder minus the volume of the one in the double cone. Write an expression for the volume of the single blueberry pancake in the sphere.

7. **Construct viable arguments.** Show that the relationship you identified in Item 6 holds true for the volumes of the entire stacks of pancakes in each figure as well.

8. Write the formula for the volume of a sphere.

9. What is the formula for the volume of a hemisphere?

10. The Blueberry Bubble will have a base pancake with a radius of 3 inches, and each of the six pancakes in the hemisphere will have a thickness of 0.5 inch.
    a. Draw a diagram for the Blueberry Bubble.
    b. What is the volume of the Blueberry Bubble?
Lesson 36-2
Volume of Spheres

Check Your Understanding

11. Suppose a sphere is divided into four equal quadrants. Write a formula to find the volume for $\frac{1}{4}$ of a sphere.

12. In terms of units of measure, explain why the surface area formula for a sphere has a squared radius and the volume formula for a sphere has a cubed radius.

13. The diameter of a sphere is 8 in. What is the volume of the sphere?

14. A sphere has a surface area of approximately $314 \text{ in.}^2$. What is the volume of the sphere to the nearest cubic inch?

LESSON 36-2 PRACTICE

15. Find the volume of the figure shown, to the nearest hundredth.

16. Determine the volume of the sphere shown below, to the nearest tenth.

17. Construct viable arguments. Four golf balls, each with radius 1 inch, fit snugly into a 2 in. × 2 in. × 8 in. box. Is the total volume left over inside the box greater than or less than the volume of a fifth golf ball? Justify your answer.

18. A spherical scoop of ice cream has been placed on top of a cone, as shown. If the ice cream melts completely, will it overflow the cone? Justify your answer.
Learning Targets:

- Compare parallelism in Euclidean and spherical geometries.
- Compare triangles in Euclidean and spherical geometries.

SUGGESTED LEARNING STRATEGIES: Activating Prior Knowledge, Visualization, Create Representations, Look for a Pattern, Use Manipulatives

Greta has used various relationships among geometric figures such as lines, angles, polygons, circles, and three-dimensional objects to create interest in mathematics at her restaurant. She, like many others, has relied on an axiomatic system as the basis of many geometric theorems; Euclidean properties are sufficient for most real-world geometric applications. Greta does realize, however, that the study of mathematics goes well beyond Euclidean geometry—as the world, unlike Greta’s pancakes, is not flat.

An understanding of spherical geometry is what allowed NASA engineers to perform the necessary calculations to get the rover to land on Mars in May 2008. Mathematicians, and curious folks like Greta, have pondered the existence of non-Euclidean geometries since the early 1800s. Research in these areas continues to be a considerable focus of study today.

Spherical geometry is one branch in the non-Euclidean field. It explores the geometric characteristics of figures on the surface of a sphere. There are significant differences in properties with this change in perspective. To understand the concepts related to spherical geometry, it is helpful to have a globe or a ball to contemplate these ideas.

1. **Use appropriate tools strategically.** Choose a point on the sphere provided by your teacher. Imagine that an ant walks on the sphere in a straight line so that it does not fall off.
   a. Trace the path that the ant would follow.
   b. Describe the path walked by the ant.
   c. Use a rubber band or masking tape to mark the path of the ant.

2. Choose a different point on the sphere. Imagine that a second ant walks in a straight line, trying to avoid the path of the first ant. Mark the path of the second ant.

3. How can the second ant choose a path that will not cross the path of the first ant?

**MATH TIP**

Great circles are the largest possible circles drawn on a sphere.
4. In Euclidean geometry, given a line and a point not on the line, there is exactly one line through the point parallel to the given line. Explain why no lines are parallel in spherical geometry.

5. In Euclidean geometry, if two lines intersect, they intersect in exactly one point. In spherical geometry, what is true about any two lines?

6. In spherical geometry, a line is defined to be a great circle of the sphere. Lines of longitude on a globe are examples of these. Why do you suppose that airlines design flight paths along spherical lines?

**Antipodal points** (also called antipodes) are two points that are diametrically opposite. A straight line through the center of the Earth connects them.

7. Locate the antipode for your city. State its location.

8. **Construct viable arguments.** Explain why most points on Earth have oceanic antipodes.

9. Choose a point on your sphere.
   a. Draw two great circles through the point.

   b. Use a protractor to measure the angles formed at the given point on the sphere by the great circles.

   c. What do you notice about the measures of the angles formed by the great circles?
10. **Model with mathematics.** Take a sphere. Choose a point $N$ on your sphere.
   a. Draw a great circle through point $N$.
   b. Draw a second great circle through point $N$ to create a **lune** with a 90° angle.
   c. Draw a great circle that does not contain point $N$ to divide the lune into two congruent regions.
   d. How many nonoverlapping sections are created by the intersections of the three great circles?

   e. Each section is a spherical triangle. Use a protractor to measure the three angles of a triangle.

   f. What is the sum of the measures of those three angles?

11. Choose any three random noncollinear points on the sphere. Draw the triangle through the points. Remember to draw the segments along a great circle.

12. Measure the three angles of the triangle and determine the sum of their measures.
13. Use the given diagram with triangle ABC.
   a. What happens to the sum of the measures of the angles in the triangle as points B and C come closer and closer together?

   b. Based on your answer in Part a, what is the least possible sum for the three angles of a spherical triangle?

14. Use the given diagram with triangle DQT.

   a. What happens to the measure of each angle in the triangle as the points move outward and approach the same great circle?

   b. Based on your answer in part a, what is the greatest possible sum for the three angles of a spherical triangle?

15. Write the possible measurements for the sum of the measures of the angles of a triangle as a compound inequality.
16. Are the lines of the Earth’s latitude considered lines in spherical geometry? Explain.

17. Locate two land-based antipodes on Earth.

18. Refer to Item 9. When two great circles intersect, how many points of intersection result? How many angles are formed at each point of intersection along with four lunes?

LESSON 36-3 PRACTICE

19. Why do you suppose Euclidean geometry is still considered to be an essential component of the high school curriculum?

20. Make use of structure. Compare and contrast Euclidean geometry and spherical geometry.
ACTIVITY 36 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 36-1

1. Find the surface area of a sphere, in terms of \( \pi \), given a radius of 2.5 in.

2. Paper lanterns are created and released each year to celebrate Jarrod's birthday. The lanterns are spherical in shape and have a diameter of 2 ft. About how much paper is needed to make one of these lanterns, assuming there is negligible overlap?
   A. 8.2 ft\(^2\)
   B. 12.57 ft\(^2\)
   C. 25.13 ft\(^2\)
   D. 50.24 ft\(^2\)

3. A large, spherical-shaped balloon is used by a local car lot to advertise the current car loan interest rate. The surface area of the balloon is 254.47 ft\(^2\). What is the radius of the balloon?

4. For cancer awareness month, a pink stripe was painted around the balloon in Item 3. The stripe is parallel to the center of the sphere, and is located at a distance of 1.5 ft from the center.
   a. What is the radius of the pink stripe?
   b. What is the circumference of the pink stripe, to the nearest tenth?

5. Consider a sphere whose diameter is 50 cm. Compute the surface area of the sphere.

6. The figure below is a cone “topped” with a hemisphere. Calculate the total surface area if the radius of the cone and hemisphere is 10 cm and the height of the cone is 24 cm.

7. The park rangers maintain bear-proof trash containers at a local campground. These containers have a cylindrical shape with a hemisphere-shaped lid, as shown below. Every spring the park rangers paint the containers. Which of the following measures would help the park rangers the most in determining how much paint is needed to paint one trash container?
   A. weight
   B. volume
   C. surface area
   D. cross-sectional area
Lesson 36-2

8. Find the volume of the figure shown, in terms of $\pi$.

9. Determine the approximate volume of one-half of the sphere shown below.

10. The volume of a sphere is 62,500 ft$^3$. What is the diameter of the sphere?
    A. 25 ft
    B. 30 ft
    C. 50 ft
    D. 141 ft

11. The cylinder shown below has been capped with a hemisphere at each end. The height of the cylinder is 10 cm. The radius of each hemisphere is 2 cm. Calculate the volume of the figure.

Lesson 36-3

12. Describe what is meant by non-Euclidean geometries.

13. Research hyperbolic geometry online. Describe the basis for this non-Euclidean geometry.

MATHEMATICAL PRACTICES
Make Sense of Problems and Persevere in Solving Them

14. A conical section has been carved into the flat surface of a hemisphere, as shown.
   
   a. What information is needed to calculate the total surface area?
   b. Write the steps for calculating the total surface area of this figure.
Changing Dimensions
Model Behavior
Lesson 37-1 Cubes and Spheres

Learning Targets:
• Describe how changes in the linear dimensions of a shape affect its perimeter, area, surface area, or volume.
• Use geometric shapes and their measures to model real-world objects.

SUGGESTED LEARNING STRATEGIES: Create Representations, Visualization, Predict and Confirm, Quickwrite

Howie Shrinkett builds scale models for museums. He was commissioned to build models of the solar system for the Hubble Planetarium, models of the Egyptian pyramids for the Ptolemy Natural History Museum, and models of the building that houses the Cubism Museum of Modern Art.

1. For the gift shop in the Cubism Museum of Modern Art, Howie created paperweights, key chains and planters. He began by investigating the dimensions, surface area, and volume of the cubes. Complete the table below.

<table>
<thead>
<tr>
<th>Cubed-Shaped Item</th>
<th>Edge Length (in inches)</th>
<th>Area of Base (in square inches)</th>
<th>Surface Area (in square inches)</th>
<th>Volume (in cubic inches)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Key chain</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Paperweight</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Planter</td>
<td>9</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Express regularity in repeated reasoning. Howie observed that as the length of an edge increases, so does the resulting surface area.
   a. Make a conjecture about how the surface area changes when the edge of a cube is tripled.

   b. Use your conjecture and the information from the table to predict the surface area of a cube whose edge is 27 inches. Then verify or revise your conjecture from part a.

3. Express regularity in repeated reasoning. Howie also observed that as the length of an edge increases, so does the resulting volume.
   a. Make a conjecture about how the volume changes when the edge of a cube is tripled.

   b. Use your conjecture and the information from the table to predict the volume for a cube whose edge is 27 inches and verify or revise your conjecture from Part a.
4. For the Hubble Planetarium, Howie created scale models of Mars. Complete the table below for Howie's spheres.

<table>
<thead>
<tr>
<th>Radius of Sphere (in feet)</th>
<th>Surface Area (in square feet)</th>
<th>Volume (in cubic feet)</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Howie observed that as the radius decreases, so does the resulting surface area.
   a. Make a conjecture about how the surface area changes when the radius is cut in half.

   b. Use your conjecture to predict the surface area of a sphere whose radius is 1.5 inches and verify or revise your conjecture from part a.

6. Howie also observed that as the radius decreases, so does the resulting volume.
   a. Make a conjecture about the way the volume changes when the radius is cut in half.

   b. Use your conjecture to predict the volume of a sphere whose radius is 1.5 inches and verify or revise your conjecture from part a.

7. **Reason abstractly.** Howie recalled that similar two-dimensional and three-dimensional figures are those that have the same shape but not necessarily the same size. Does this imply that all cubes are similar to all other cubes, and that all spheres are similar to all other spheres? Explain below.

**MATH TIP**

The surface area of a sphere is \(4\pi r^2\), and the volume is \(\frac{4}{3}\pi r^3\).
Lesson 37-1
Cubes and Spheres

Check Your Understanding

8. The edge length of a cube is 4 feet.
   a. Find the surface area of the cube.
   b. Predict the surface area if the edge length of the cube is tripled.
   c. By what factor does the surface area increase?
   d. Suppose the edge length is quadrupled. By what factor will the surface area increase?
   e. Describe any pattern you notice between surface area and the scale factor of the edge length.

9. The radius of a sphere is 9 feet.
   a. Find the volume of the sphere, in terms of \( \pi \).
   b. Predict the volume if the radius of the sphere is tripled.
   c. By what factor does the volume increase?
   d. Suppose the radius is quadrupled. By what factor will the volume increase?
   e. Describe any pattern you notice between volume and the scale factor of the radius.

10. Suppose the values in the table for the sphere had been given in yards. Explain how your conjectures in Items 5a and 6a would change.

LESSEE 37-1 PRACTICE

11. One of the cubical display tables in the gift shop of a museum has side lengths of 6 feet. Howie wants to purchase a new table, but he wants each side to be reduced by 25%.
   a. What is the surface area of the new table?
   b. What is the volume of the new table?

12. A sphere with diameter 5 feet is being painted. How much more paint would be needed to paint a sphere that has four times the diameter?

13. Suppose the diameter of the sphere in Item 12 is tripled. What effect would that have on the volume of the sphere? Express your answer as an exponent.

14. Kendrick purchases a refrigerator for his college dorm. The sides of the refrigerator are congruent. The volume of the refrigerator is 5.832 cubic feet. The refrigerator in the dorm kitchen is a similar shape but has a volume of 157.464 cubic feet.
   a. Write the simplified ratio of the side length of the smaller refrigerator to that of the larger refrigerator.
   b. Write the simplified ratio of the surface area of the smaller refrigerator to that of the larger refrigerator.

15. Critique the reasoning of others. Buoys are used in the ocean as markers. The spherical-shaped markers in current use at a beach have a radius of 2 meters. One of the lifeguards stated that if the radius of the buoys were doubled, the buoys would have twice as much surface area and would be easier to see. Do you agree with the lifeguard's statement? Explain.
Learning Targets:

- Describe how changes in the linear dimensions of a shape affect its perimeter, area, surface area, or volume.
- Use geometric shapes and their measures to model real-world objects.

SUGGESTED LEARNING STRATEGIES: Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Interactive Word Wall

As Howie begins his pyramid project for the Ptolemy Museum, he realizes that he needs more than one dimension to determine a scale model. The scale models must be geometrically similar.

1. **Make use of structure.** Which of the pyramids below are similar? Explain.

![Pyramids](image)

Pyramid 1

Pyramid 2

Pyramid 3

2. Howie has built a square pyramid. The base edge is 10 centimeters, and the height is 12 centimeters. He knows that he needs the height of the second pyramid to be 24 centimeters. Determine the scale factor of the small pyramid to that of the larger pyramid.

3. Determine the length of the base edge of the larger pyramid.
Lesson 37-2
Pyramids and Cylinders

4. Complete the table below for Howie's pyramids.

<table>
<thead>
<tr>
<th>Height of Pyramid (in.)</th>
<th>Length of Base Edge (in.)</th>
<th>Slant Height (in.)</th>
<th>Perimeter of the Base (in.)</th>
<th>Area of the Base (sq. in.)</th>
<th>Lateral Area (sq. in.)</th>
<th>Surface Area (sq. in.)</th>
<th>Volume (cu. in.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pyramid 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pyramid 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratio of Pyramid 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pyramid 2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

5. Howie also designs models for cylindrical drums as shown below. Complete the table below, and justify whether or not the two cylinders are similar.

![Cylinders](image)

<table>
<thead>
<tr>
<th>Radius (cm)</th>
<th>Circumference of Base (cm)</th>
<th>Surface Area (square cm)</th>
<th>Volume (cubic cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cylinder 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cylinder 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ratio of Cylinder 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cylinder 2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
6. **Express regularity in repeated reasoning.** Complete the table below for any pair of similar solids with the scale factor \( \frac{a}{b} \).

<table>
<thead>
<tr>
<th>Ratio of Sides/Edges or Other Corresponding Measurements</th>
<th>Ratio of Perimeters</th>
<th>Ratio of Areas</th>
<th>Ratio of Volumes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Similarity Ratios</td>
<td>( \frac{a}{b} )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. Write a conjecture about the ratio of corresponding perimeters in any pair of similar solids.

b. Write a conjecture about the ratio of corresponding areas in any pair of similar solids.

c. Write a conjecture about the ratio of the volumes of any similar solid.

7. **Model with mathematics.** Draw and label two solids whose corresponding dimensions do not have the same scale factors. Based on the conjectures you made in Item 6, verify that the two figures are not similar to one another.
Lesson 37-2
Pyramids and Cylinders

8. A tetrahedron has four faces, each of which is an equilateral triangle. The diagram below shows two tetrahedrons. Suppose the tetrahedron on the left has 6-inch edges, and the tetrahedron on the right has 8-inch edges.

![Diagram of two tetrahedrons]

a. What is the ratio of the surface area of the smaller tetrahedron to the surface area of the larger tetrahedron?

b. What is the ratio of the volume of the smaller tetrahedron to the volume of the larger tetrahedron?

9. **Reason quantitatively.** Two similar cones have volumes of $125\pi$ in.$^3$ and $64\pi$ in.$^3$.
   a. Determine the scale factor for these cones.

   b. If the total area of the larger cone is $75\pi$ in.$^2$, calculate the total area of the smaller cone.
Check Your Understanding

10. In Item 8a, write the factor you used to determine the ratio of surface area with an exponent. Then write the factor you used in Item 8b to determine the ratio of volume with an exponent.

11. True or False? If the radius of a right cylinder is increased by a factor of 3, and the height stays the same, the volume of the cylinder will increase by a factor of 3. Explain your reasoning.

LESSON 37-2 PRACTICE

12. Make sense of problems. A small can of coffee is 3 in. tall with a 2 in. radius. It sells for $6.26. A larger can of coffee is 9 in. tall with a 6 in. radius. It sells for $12.52. Is the larger can of coffee priced proportionally in regard to the volume of the smaller can? Explain.

13. The radius and height of the cylinder shown is reduced by 75%. Describe the effect on the surface area.

14. If the surface areas of two similar cylinders are $10\pi \text{ ft}^2$ and $160\pi \text{ ft}^2$, what is the ratio of the volume of the smaller cylinder to the volume of the larger cylinder?

15. Attend to precision. Describe the effect on the volume of a square pyramid if its height is doubled.
ACTIVITY 37 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 37-1

1. If the surface area of two similar spheres is \(256\pi \text{ ft}^2\) and \(576\pi \text{ ft}^2\), what is the ratio of the volume of the smaller sphere to the volume of the larger sphere?

2. A modeling platform is in the shape of a cube. It has a surface area of 294 \(\text{ft}^2\). Which of the following describes what happens to the surface area when the dimensions of the cube are doubled.
   A. The surface area is halved.
   B. The surface area stays the same.
   C. The surface area doubles.
   D. The surface area quadruples.

3. A cube has a volume of 1331 \(\text{in}^3\). What are the approximate dimensions of the cube if the volume is increased by 1.5?

4. A packaging designer created a new container using two similar, three-dimensional figures. The scale used to create the figures was 2 : 5. Which of the following is the ratio of their volumes?
   A. 2 : 5
   B. 4 : 10
   C. 4 : 25
   D. 8 : 125

5. A sphere has a volume of \(457 \frac{1}{3} \pi \text{ m}^3\). The radius is halved.
   a. What is the new radius?
   b. What is the new volume of the sphere?

Lesson 37-2

6. A cylinder has a radius of 4 cm and a height of 9 cm. A similar cylinder has a radius of 6 cm.
   a. Find the scale factor of the smaller cylinder to the larger cylinder
   b. What is the ratio of the circumferences of the bases?
   c. What is the ratio of the lateral areas of the cylinders?
   d. What is the ratio of the volumes of the cylinders?
   e. If the volume of the smaller cylinder is \(144\pi \text{ cm}^3\), what is the volume of the larger cylinder?

7. You purchased a scale model of the Eiffel Tower. The scale factor is 1 : 1180. The model is 10 inches high and has a surface area of 1.7 \(\text{ft}^2\).
   a. What is the height of the Eiffel Tower?
   b. What is the surface area of the Eiffel Tower?
8. The volume of the cylinder, shown by the net below, needs to increase.
   a. What is the volume of the cylinder?
   b. What is the volume of the cylinder if the radius and height are doubled?
   c. Write the ratio of the volume of the current cylinder to the volume of the new cylinder if the measures of the current cylinder are doubled.

9. A pyramid has a height of 6 cm and base edge 8 cm, as shown.

   A pyramid similar to the one above has a height of 9 cm.
   a. Find the scale factor of the smaller pyramid to the larger pyramid.
   b. What is the ratio of the base edges?
   c. Find the base edge of the larger pyramid.
   d. What is the ratio of the lateral areas?
   e. What is the ratio of the volumes?
   f. If the volume of the larger pyramid is 130.5 cm$^3$, what is the volume of the smaller pyramid?

10. The square base of a pyramid has side lengths $x$.

   a. What is the effect on the volume of the square pyramid if the sides change to $2x$?
   b. What is the effect on the volume of the square pyramid if the sides change to $\frac{1}{2}x$?

11. You purchased a scale model of the Statue of Liberty. The scale factor is 1 : 456. The model is 8 inches high and has a surface area of 0.052 ft$^2$.
   a. What is the height of the Statue of Liberty?
   b. What is the surface area of the Statue of Liberty?

12. The diameter of an oblique cylinder is 32 cm and the height is 4 cm. What is the volume of a right cylinder if the dimensions are halved?
   A. $8\pi$ cm$^3$
   B. $128\pi$ cm$^3$
   C. $512\pi$ cm$^3$
   D. $1024\pi$ cm$^3$

MATHEMATICAL PRACTICES
Look For and Make Use of Structure

13. The ratio of the edge lengths of two given cubes is $\frac{7}{2}$.
   a. Describe how to find the ratio for the area of the faces of the cubes.
   b. Describe how to find the ratio for the volumes of the cubes.
Storage vessels containing organic liquids should be designed and built according to the American Petroleum Institute API-650 specification. Storage vessels come in various shapes and sizes, but in general most are installed within a containment basin for safety reasons. Some vessels need to be pressurized if they are being used to store organic liquids with high vapor pressures. Typically, the best design for this application, however costly, is a sphere.

A petroleum company plans to purchase some spherical pressurized storage vessels. The process-engineering group and the purchasing department are working together to analyze possible options.

1. Complete the table based on the dimensions provided.

<table>
<thead>
<tr>
<th>Spherical Vessel</th>
<th>Diameter</th>
<th>Surface Area</th>
<th>Volume</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vessel A</td>
<td>20 m</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Vessel B</td>
<td></td>
<td></td>
<td>33,510.3 m³</td>
</tr>
<tr>
<td>Vessel C</td>
<td></td>
<td>11,309.7 m²</td>
<td>11,309.7 m²</td>
</tr>
</tbody>
</table>

2. The company plans to have the tanks painted after they are installed. Describe the effect that doubling the diameter has on the amount of paint needed to paint the tanks. What happens when the diameter is increased by a factor of 3?

3. Describe the effect that doubling the diameter has on the capacity of a spherical storage vessel. What happens to the volume when the diameter is tripled?

4. How would the amount of paint needed and the capacity change if the company chose a storage vessel with a diameter that was half as large as the diameter of Vessel A? Explain your reasoning.

The supports for the tanks are in the shapes of cylinders. The supports for Vessel A and Vessel C are similar. Each support for Vessel A has volume $450,000 \pi \text{ cm}^3$ and surface area $60,450 \pi \text{ cm}^2$.

5. If the volume of each cylindrical support for Vessel C is $1,518,750 \pi \text{ cm}^3$, write a proportion that can be used to determine the surface area of each of Vessel C’s supports. Explain how you arrived at your answer.
## Scoring Guide

<table>
<thead>
<tr>
<th>Mathematics Knowledge and Thinking (Items 1, 2, 3, 4, 5)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clear and accurate understanding of surface area and volume of a sphere</td>
<td>Mostly correct understanding of surface area and volume of a sphere</td>
<td>Partially correct understanding of surface area and volume of a sphere</td>
<td>Little or no understanding of surface area and volume of a sphere</td>
<td></td>
</tr>
<tr>
<td>Clear and accurate understanding of the relationships between scale factor, ratio of areas, and ratio of volumes of similar figures</td>
<td>Mostly correct understanding of the relationships between scale factor, ratio of areas, and ratio of volumes of similar figures</td>
<td>Partially correct understanding of the relationships between scale factor, ratio of areas, and ratio of volumes of similar figures</td>
<td>Little or no understanding of the relationships between scale factor, ratio of areas, and ratio of volumes of similar figures</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem Solving (Items 1, 2, 3, 4)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>An appropriate and efficient strategy that results in correct answers</td>
<td>A strategy that may include unnecessary steps but results in mostly correct answers</td>
<td>A strategy that results in some incorrect answers</td>
<td>No clear strategy when solving problems</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Mathematical Modeling / Representations (Item 5)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Clear and accurate understanding of creating a proportion to solve for the surface area of Vessel C’s supports</td>
<td>Mostly accurate understanding of creating a proportion to solve for the surface area of Vessel C’s supports</td>
<td>Partial understanding of creating a proportion to solve for the surface area of Vessel C’s supports</td>
<td>Little or no understanding of creating a proportion to solve for the surface area of Vessel C’s supports</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reasoning and Communication (Items 4, 5)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>Precise use of appropriate mathematics and language to explain how the volume and surface area were affected by the dimension change</td>
<td>Mostly correct use of appropriate mathematics and language to explain how the volume and surface area were affected by the dimension change</td>
<td>Misleading or confusing use of appropriate mathematics and language to explain how the volume and surface area were affected by the dimension change</td>
<td>Incomplete or inaccurate use of appropriate mathematics and language to explain how the volume and surface area were affected by the dimension change</td>
<td></td>
</tr>
<tr>
<td>Precise use of appropriate mathematics and language to explain how the proportion is written</td>
<td>Mostly correct use of appropriate mathematics and language to explain how the proportion is written</td>
<td>Misleading or confusing use of appropriate mathematics and language to explain how the proportion is written</td>
<td>Incomplete or inaccurate use of appropriate mathematics and language to explain how the proportion is written</td>
<td></td>
</tr>
</tbody>
</table>

The solution demonstrates these characteristics: