In this unit, students write the equations of quadratic functions to model situations and then graph these functions. They study methods of finding solutions to quadratic equations and interpreting these solutions. In the process, students learn about complex numbers.

Vocabulary Development

The key terms for this unit can be found on the Unit Opener page. These terms are divided into Academic Vocabulary and Math Terms. Academic Vocabulary includes terms that have additional meaning outside of math. These terms are listed separately to help students transition from their current understanding of a term to its meaning as a mathematics term. To help students learn new vocabulary:

- Have students discuss meaning and use graphic organizers to record their understanding of new words.
- Remind students to place their graphic organizers in their math notebooks and revisit their notes as their understanding of vocabulary grows.
- As needed, pronounce new words and place pronunciation guides and definitions on the class Word Wall.

Embedded Assessments

Embedded Assessments allow students to do the following:

- Demonstrate their understanding of new concepts.
- Integrate previous and new knowledge by solving real-world problems presented in new settings.

They also provide formative information to help you adjust instruction to meet your students’ learning needs.

Prior to beginning instruction, have students unpack the first Embedded Assessment in the unit to identify the skills and knowledge necessary for successful completion of that assessment. Help students create a visual display of the unpacked assessment and post it in your class. As students learn new knowledge and skills, remind them that they will be expected to apply that knowledge to the assessment. After students complete each Embedded Assessment, turn to the next one in the unit and repeat the process of unpacking that assessment with students.
Additional Resources
Additional resources that you may find helpful for your instruction include the following, which may be found in the Teacher Resources at SpringBoard Digital.

- Unit Practice (additional problems for each activity)
- Getting Ready Practice (additional lessons and practice problems for the prerequisite skills)
- Mini-Lessons (instructional support for concepts related to lesson content)
Unit Overview
This unit focuses on quadratic functions and equations. You will write the equations of quadratic functions to model situations. You will also graph quadratic functions and other parabolas and interpret key features of the graphs. In addition, you will study methods of finding solutions of quadratic equations and interpreting the meaning of the solutions. You will also extend your knowledge of number systems to the complex numbers.

Key Terms
As you study this unit, add these and other terms to your math notebook. Include in your notes your prior knowledge of each word, as well as your experiences in using the word in different mathematical examples. If needed, ask for help in pronouncing new words and add information on pronunciation to your math notebook. It is important that you learn new terms and use them correctly in your class discussions and in your problem solutions.

Academic Vocabulary
- justify
- derive
- verify
- advantage
- disadvantage
- counterexample

Math Terms
- quadratic equation
- standard form of a quadratic equation
- imaginary number
- complex number
- complex conjugate
- completing the square
- discriminant
- root
- zero
- parabola
- focus
- directrix
- axis of symmetry
- vertex
- quadratic regression
- vertex form

Essential Questions
How can you determine key attributes of a quadratic function from an equation or graph?
How do graphic, symbolic, and numeric methods of solving quadratic equations compare to one another?

This unit has three embedded assessments, following Activities 9, 11, and 13. By completing these embedded assessments, you will demonstrate your understanding of key features of quadratic functions and parabolas, solutions to quadratic equations, and systems that include nonlinear equations.

- Embedded Assessment 1: Applications of Quadratic Functions and Equations p. 151
- Embedded Assessment 2: Writing and Transforming Quadratic Functions p. 191
- Embedded Assessment 3: Graphing Quadratic Functions and Solving Systems p. 223

Developing Math Language
As this unit progresses, help students make the transition from general words they may already know (the Academic Vocabulary) to the meanings of those words in mathematics. You may want students to work in pairs or small groups to facilitate discussion and to build confidence and fluency as they internalize new language. Ask students to discuss new academic and mathematics terms as they are introduced, identifying meaning as well as pronunciation and common usage. Remind students to use their math notebooks to record their understanding of new terms and concepts.

As needed, pronounce new terms clearly and monitor students' use of words in their discussions to ensure that they are using terms correctly. Encourage students to practice fluency with new words as they gain greater understanding of mathematical and other terms.
Write your answers on notebook paper. Show your work.

Factor the expressions in Items 1–4 completely.

1. $6x^3y + 12x^2y^2$
2. $x^2 + 3x - 40$
3. $x^2 - 49$
4. $x^2 - 6x + 9$

5. Graph $f(x) = \frac{3}{4}x - \frac{3}{2}$.

6. Graph a line that has an $x$-intercept of 5 and a $y$-intercept of $-2$.

7. Graph $y = |x|$, $y = |x + 3|$, and $y = |x| + 3$ on the same grid.

8. Solve $x^2 - 3x - 5 = 0$.

Getting Ready Practice

For students who may need additional instruction on one or more of the prerequisite skills for this unit, Getting Ready practice pages are available in the Teacher Resources at SpringBoard Digital. These practice pages include worked-out examples as well as multiple opportunities for students to apply concepts learned.
Fence Me In is a business that specializes in building fenced enclosures. One client has purchased 100 ft of fencing to enclose the largest possible rectangular area in her yard.

Work with your group on Items 1–7. As you share ideas, be sure to explain your thoughts using precise language and specific details to help group members understand your ideas and your reasoning.

1. If the width of the rectangular enclosure is 20 ft, what must be the length? Find the area of this rectangular enclosure.

The length would be 30 ft and the area of the enclosure would be 600 ft².

2. Choose several values for the width of a rectangle with a perimeter of 100 ft. Determine the corresponding length and area of each rectangle. Share your values with members of your class. Then record each set of values in the table below.

<table>
<thead>
<tr>
<th>Width (ft)</th>
<th>Length (ft)</th>
<th>Area (ft²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>49</td>
<td>49</td>
</tr>
<tr>
<td>5</td>
<td>45</td>
<td>225</td>
</tr>
<tr>
<td>10</td>
<td>40</td>
<td>400</td>
</tr>
<tr>
<td>20</td>
<td>30</td>
<td>600</td>
</tr>
<tr>
<td>25</td>
<td>25</td>
<td>625</td>
</tr>
<tr>
<td>30</td>
<td>20</td>
<td>600</td>
</tr>
<tr>
<td>40</td>
<td>10</td>
<td>400</td>
</tr>
<tr>
<td>45</td>
<td>5</td>
<td>225</td>
</tr>
</tbody>
</table>

3. Make sense of problems. What is the relationship between the length and width of a rectangle with perimeter of 100 ft?

The length plus width equals 50 feet ($l + w = 50$).

4. Based on your observations, predict if it is possible for a rectangle with perimeter of 100 ft to have each area. Explain your reasoning.

Explanations will vary; answers and sample responses follow.

a. 400 ft²

Yes. Dimensions of 10 ft and 40 ft have an area of 400 ft².

b. 500 ft²

Yes. Dimensions of 10 ft and 40 ft have an area of 400 ft²; dimensions of 20 ft and 30 ft have an area of 600 ft². There will be intermediate dimensions that have an area of 500 ft².

A canoe livery rents canoes for a flat fee of $30, plus an additional $10 per hour.

$c(h) = 30 + 10h$

Have students find the amount it costs to rent the canoe for 4 hours.

$c(4) = 40 + (10)(4) = 80$

1–2 Activating Prior Knowledge, Group Presentation Item 1 allows students to review length, width, and area of rectangles. For Item 2, it is essential that students gather adequate amounts of data to establish patterns.

3–4 Look for a Pattern, Guess and Check The fact that the length and width have a sum of 50 becomes apparent by inspecting the first two columns of the table. Some students may express this in other ways. For example, the length equals 50 feet minus the width. Students will likely answer Item 4 from an inductive viewpoint based on the values in their tables. Most groups will have an area of 400 ft², no groups will have 500 ft² (but will guess it is possible because they have values larger than 500 ft² in their table); and no groups will have 700 ft². Value all predictions at this time as reasonable guesses.

Common Core State Standards for Activity 7

HSA-CED.A.1 Create equations and inequalities in one variable and use them to solve problems.

HSA-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.

HSA-SSE.A.1 Interpret expressions that represent a quantity in terms of its context.

HSA-SSE.A.1a Interpret parts of an expression, such as terms, factors, and coefficients.

HSF-IF.B.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

HSF-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.
5–6 Create Representations  The patterns students recognize in Item 3 are used to create algebraic representations for the context. It is essential that students understand that the relationship between length and width is necessary to create a function with one independent variable.

7–8 Create Representations  Students should graph this function by choosing $x$-values that result in $y$-values that are easy to plot. Students can also use their data from Item 2 to plot points. If students remember properties of quadratic functions from Algebra 1, they may use them to graph the function.

9 Activating Prior Knowledge, Debriefing  Remind students when going from the $x$- and $y$-values of their graphing calculators to use $A(l)$ to represent the function ($y$) and $l$ to represent the length ($x$). For additional technology resources, visit SpringBoard Digital. Think about what this graph actually represents. It represents the area of a rectangle, based upon its length. If necessary, review inequalities, interval notation, and set notation. Emphasize that the inequality sign is $<$ and not $\leq$. Students should be prepared to explain why. Also note how this affects the interval notation brackets (rounded rather than square).

8. Use appropriate tools strategically. Now use a graphing calculator to graph the quadratic function $A(l)$. Set your window to correspond to the values on the axes on the graph in Item 7.

Check students’ work.

9. Use the function $A(l)$ and your graphs from Items 7 and 8 to complete the following.

a. What is the reasonable domain of the function in this situation? Express the domain as an inequality, in interval notation, and in set notation.

$$0 < l < 50; \{l | l \in \mathbb{R}, 0 < l < 50\}$$
Lesson 7–1
Analyzing a Quadratic Function

b. Over what interval of the domain is the value of the function increasing? Over what interval of the domain is the value of the function decreasing?
The function increases for $0 < \ell \leq 25$ and decreases for $25 \leq \ell < 50$.

10. What is the maximum rectangular area that can be enclosed by 100 ft of fencing? Justify your answer.
625 ft². Sample justification: The graph shows that the maximum value of the function occurs when $\ell = 25$, and $A(25) = -25^2 + 50(25) = 625$.

11. a. What is the reasonable range of $A(\ell)$ in this situation? Express the range as an inequality, in interval notation, and in set notation.
$0 < A \leq 625; (0, 625); \{A \mid A \in \mathbb{R}, 0 < A \leq 625\}$

b. Explain how your answer to Item 10 helped you determine the reasonable range.
The maximum value of the function is the same as the maximum value of the range.

12. Reason quantitatively. Revise or confirm your predictions from Item 4. If a rectangle is possible, estimate its dimensions and explain your reasoning. Review the draft of your revised or confirmed predictions. Be sure to check that you have included specific details, the correct mathematical terms to support your explanations, and that your sentences are complete and grammatically correct. You may want to pair-share with another student to critique each other’s drafts and make improvements.
a. 400 ft²
The graph of $A(\ell)$ shows that there are two rectangles that will have this area, a 10 ft $\times$ 40 ft rectangle and a 40 ft $\times$ 10 ft rectangle.

b. 500 ft²
The graph of $A(\ell)$ shows two possible lengths that will yield an area of 500 ft². The lengths are not easily determined from the graph; however, the points of intersection appear to be around 14 ft and 36 ft. There are two rectangles that will have this area, an approximately 14 ft $\times$ 36 ft rectangle and an approximately 36 ft $\times$ 14 ft rectangle.

c. 700 ft²
Since the graph of $A(\ell)$ never reaches $A(\ell) = 700$, an area of 700 ft² is not possible.

MINI-LESSON: Quadratic Formula
If students need additional help with using the quadratic formula, a mini-lesson is available to provide practice.
See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.
ACTIVITY 7 Continued

13–14 Quickwrite, Self Revision, Peer Revision, Debriefing. Many students will expect the result to be a square. Some students may disagree, however, mistakenly believing that a square is not a rectangle. Appropriate instruction regarding quadrilaterals and how to reason the answer logically from the definitions of square and rectangle may be necessary.

Check Your Understanding
Debrief students’ answers to these items to ensure that they understand concepts related to quadratic functions and to solving quadratic equations by graphing.

Answers
15. Sample answers: The function \( A(l) = -l^2 + 50l \) is quadratic, because it can be written in the form \( f(x) = ax^2 + bx + c \), with \( a = -1 \), \( b = 50 \), and \( c = 0 \). The function is quadratic, because its graph is a parabola.
16. Sample answer: The graph of a linear function is a line, and the graph of a quadratic function is a parabola. The graph of a linear function has at most one x-intercept, but the graph of a quadratic function can have two x-intercepts. The graph of a quadratic function has a maximum or minimum value, but the graph of a linear function does not.
17. No. A quadratic function has a maximum or a minimum value. If it has a maximum, its range does not include values greater than the maximum. If it has a minimum, its range does not include values less than the minimum.
18. Graph the function \( f(x) = x^2 + 2x \). Then find the points on the graph where \( f(x) = 3 \). The x-coordinates of these points are the solutions of the quadratic equation \( x^2 + 2x = 3 \).

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand how to write an equation that represents the area of a rectangle. For those students requiring additional practice, have them create problems similar to Lesson Practice Items 19–21 to swap with other classmates.

LESSON 7-1 PRACTICE

13. What are the length and width of the largest rectangular area that can be enclosed by 100 ft of fencing?
   25 ft by 25 ft

14. The length you gave in Item 13 is the solution of a quadratic equation in terms of \( l \). Write this equation. Explain how you arrived at this equation. **625 = -l^2 + 50l; I found the value of \( l \) for which \( A(l) = 625 \), so I solved the equation 625 = -\( l^2 + 50\ell \).**

Check Your Understanding

15. Explain why the function \( A(l) \) that you used in this lesson is a quadratic function.
16. How does the graph of a quadratic function differ from the graph of a linear function?
17. Can the range of a quadratic function be all real numbers? Explain.
18. Explain how you could solve the quadratic equation \( x^2 + 2x = 3 \) by graphing the function \( f(x) = x^2 + 2x \).

LESSON 7-1 PRACTICE

For Items 19–21, consider a rectangle that has a perimeter of 120 ft.

19. Write a function \( B(l) \) that represents the area of the rectangle with length \( l \).
20. Graph the function \( B(l) \), using a graphing calculator. Then copy it on your paper, labeling axes and using an appropriate scale.
21. Use the graph of \( B(l) \) to find the dimensions of the rectangle with a perimeter of 120 feet that has each area. Explain your answer.
   a. 500 ft²
   b. 700 ft²
22. Critique the reasoning of others. An area of 1000 ft² is not possible. Explain why this is true.
23. How is the maximum value of a function shown on the graph of the function? How would a minimum value be shown?

21. a. 10 ft × 50 ft or 50 ft × 10 ft
   b. The intersection points of \( B(l) \) and \( B(l) = 700 \) appear to be around (16, 700) and (44, 700). The dimensions are approximately 16 ft × 44 ft and 44 ft × 16 ft.
22. Sample answer: The graph shows that the maximum rectangular area is 900 ft².
23. The maximum value is represented by the y-coordinate of the highest point on the graph. The minimum value would be represented by the y-coordinate of the lowest point.
Lesson 7-2
Factoring Quadratic Expressions

Learning Targets:
- Factor quadratic expressions of the form \( x^2 + bx + c \).
- Factor quadratic expressions of the form \( ax^2 + bx + c \).

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Vocabulary Organizer, Marking the Text, Guess and Check, Work Backward, RAFT

In the previous lesson, you used the function \( A(l) = -l^2 + 50l \) to model the area in square feet of a rectangle that can be enclosed with 100 ft of fencing.

1. **Reason quantitatively.** What are the dimensions of the rectangle if its area is 525 ft\(^2\)? Explain how you determined your answer.

2. One way to find the dimensions of the rectangle is to solve a quadratic equation algebraically. What quadratic equation could you have solved to answer Item 1?

   \[ 525 = -l^2 + 50l \]

3. Write the quadratic equation from Item 2 in the form \( a\ell^2 + bl + c = 0 \), where \( a > 0 \). Give the values of \( a \), \( b \), and \( c \).

   \[ \ell^2 - 50\ell + 525 = 0; \quad a = 1, \ b = -50, \ c = 525 \]

As you have seen, graphing is one way to solve a quadratic equation. However, you can also solve quadratic equations algebraically by factoring.

You can use the graphic organizer shown in Example A on the next page to recall factoring trinomials of the form \( x^2 + bx + c = 0 \). Later in this activity, you will solve the quadratic equation from Item 3 by factoring.

**MATH TERMS**

A quadratic equation can be written in the form \( ax^2 + bx + c = 0 \), where \( a \neq 0 \). An expression in the form \( ax^2 + bx + c \), \( a \neq 0 \), is a quadratic expression.

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**Activity 7 • Applications of Quadratic Functions**

**Lesson 7-2**

**PLAN**

**Pacing:** 1 class period

**Chunking the Lesson**

#1–3 Example A

#4 Example B

Check Your Understanding

Lesson Practice

**TEACH**

**Bell-Ringer Activity**

Have students factor the following polynomials completely.

1. \( x^2 - 9 \) \quad \( \quad (x + 3)(x - 3) \)

2. \( f^2 + 9f + 8 \) \quad \( \quad (f + 1)(f + 8) \)

3. \( y^3 + y^2 - 6y \) \quad \( \quad (y + 3)(y - 2) \)

Discuss that in this Bell-Ringer Activity, Item 1 is a difference of two squares, Item 2 is a trinomial, and Item 3 has a common factor of \( y \) and a trinomial that factors.

**1–3 Chunking the Activity, Discussion Groups, Debriefing** For these items, have students work with a partner or in small groups. In Item 1, encourage students to trace the graph of \( A(l) = -l^2 + 50l \) or to use the table function on their graphing calculators to find the two corresponding values of \( l \) when \( A(l) = 525 \). In Items 2 and 3, students will find that this problem could also be solved without the benefit of a graph by writing and solving (by factoring) the quadratic equation represented by this function.

**Developing Math Language**

Review with students the difference between factoring an expression and solving an equation. In earlier courses, students spend a great deal of time factoring quadratic expressions. Students then move on to solving quadratic equations by factoring, by taking a quadratic expression that is set equal to zero and finding solutions that make a true sentence.
Example A

Factor \( x^2 + 12x + 32 \).

Step 1: Place \( x^2 \) in the upper left box and the constant term 32 in the lower right.

\[
\begin{array}{c|c}
\hline
x^2 & 32 \\
\hline
\end{array}
\]

Step 2: List factor pairs of 32, the constant term. Choose the pair that has a sum equal to 12, the coefficient \( b \) of the \( x \)-term.

<table>
<thead>
<tr>
<th>Factors</th>
<th>Sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>32 1</td>
<td>32 + 1 = 33</td>
</tr>
<tr>
<td>16 2</td>
<td>16 + 2 = 18</td>
</tr>
<tr>
<td>8 4</td>
<td>8 + 4 = 12</td>
</tr>
</tbody>
</table>

Step 3: Write each factor as coefficients of \( x \) and place them in the two empty boxes. Write common factors from each row to the left and common factors for each column above.

\[
\begin{array}{c|c|c|c}
\hline
x^2 & x & 8x \\
\hline
x & 8 & 8x \\
4 & 4x & 32 \\
\hline
\end{array}
\]

Step 4: Write the sum of the common factors as binomials. Then write the factors as a product.

Solution: \( x^2 + 12x + 32 = (x + 4)(x + 8) \)

Try These A

a. Factor \( x^2 - 7x + 12 \), using the graphic organizer. Then check by multiplying.

\[
\begin{array}{c|c|c|c}
\hline
x & -3 \\
\hline
x^2 & -3x \\
-4 & -4x & 12 \\
\hline
\end{array}
\]

Factor, and then check by multiplying. Show your work.

b. \( x^2 + 9x + 14 \)  
   \( (x + 7)(x + 2) \)

c. \( x^2 - 7x - 30 \)  
   \( (x - 10)(x + 3) \)

d. \( x^2 - 12x + 36 \)  
   \( (x - 6)^2 \) or \( (x - 6)(x - 6) \)

e. \( x^2 - 144 \)  
   \( (x + 12)(x - 12) \)

f. \( 5x^2 + 40x + 75 \)  
   \( 5(x + 3)(x + 5) \)

g. \( -12x^2 + 108 \)  
   \( -12(x + 3)(x - 3) \)

MATH TIP

A difference of squares \( a^2 - b^2 \) is equal to \((a - b)(a + b)\). A perfect square trinomial \( a^2 + 2ab + b^2 \) is equal to \((a + b)^2\).
Lesson 7-2
Factoring Quadratic Expressions

Before factoring quadratic expressions $ax^2 + bx + c$, where the leading coefficient $a \neq 1$, consider how multiplying binomial factors results in that form of a quadratic expression.

4. **Make sense of problems.** Use a graphic organizer to multiply $(2x + 3)(4x + 5)$.
   a. Complete the graphic organizer by filling in the two empty boxes.
   b. $(2x + 3)(4x + 5) = 8x^2 + \underline{10x} + \underline{12x} + 15$
      
      
      
      
      $ = 8x^2 + \underline{22x} + 15$

Using the Distributive Property, you can see the relationship between the numbers in the binomial factors and the terms of the trinomial.

To factor a quadratic expression $ax^2 + bx + c$, work backward from the coefficients of the terms.

**Example B**
Factor $6x^2 + 13x - 5$. Use a table to organize your work.

- **Step 1:** Identify the factors of 6, which is $a$, the coefficient of the $x^2$-term.
- **Step 2:** Identify the factors of $-5$, which is $c$, the constant term.
- **Step 3:** Find the numbers whose products add together to equal 13, which is $b$, the coefficient of the $x$-term.
- **Step 4:** Then write the binomial factors.

<table>
<thead>
<tr>
<th>Factors of 6</th>
<th>Factors of $-5$</th>
<th>Sum = 13?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 and 6</td>
<td>$-1$ and $5$</td>
<td>$1(5) + 6(-1) = -1$</td>
</tr>
<tr>
<td>1 and 6</td>
<td>5 and $-1$</td>
<td>$1(-1) + 6(5) = 29$</td>
</tr>
<tr>
<td>2 and 3</td>
<td>$-1$ and $5$</td>
<td>$2(5) + 3(-1) = 7$</td>
</tr>
<tr>
<td>2 and 3</td>
<td>5 and $-1$</td>
<td>$2(-1) + 3(5) = 13$ ✔</td>
</tr>
</tbody>
</table>

**Solution:** $6x^2 + 13x - 5 = (2x + 5)(3x - 1)$

**MATH TIP**
Check your answer by multiplying the two binomials.

$(2x + 5)(3x - 1)$

$= 6x^2 - 2x + 15x - 5$

$= 6x^2 + 13x - 5$

**ACTIVITY 7 Continued**

4. **Activating Prior Knowledge** This item extends the concept to trinomials of the form $ax^2 + bx + c$, where $a \neq 1$. Students first investigate multiplying binomials using a graphic organizer and then use the organizer to see how the terms of the trinomial relate to the terms in the binomial factors.

**Example B Marking the Text, Summarizing, Work Backward** In this Example, students move from the visual (graphic organizer) to the abstract method of factoring trinomials with a leading coefficient not equal to 1.

Note that the table in Step 4 does not list all possible combinations of factors of 6 and factors of $-5$. Other combinations include $-1, -6$ and $-1, 5; -1, -6$ and $1, -5; -2, -3$ and $-1, 5; and -2, -3$ and $1, -5$. When solving these types of problems, students may need to check many combinations before finding the correct binomial factors.

**Teacher to Teacher**
Note that in a trinomial such as $6x^2 + 13x - 5$, the pairs of factors $(-1 \text{ and } -6)$, as well as $(-2 \text{ and } -3)$, could be used to represent the factors of 6. While it is much easier to use positive coefficients for the first terms, it should be noted that other factorizations are possible. For instance: $6x^2 + 13x - 5$ can also be factored as $(2x + 5)(3x - 1)$. Notice that every sign in the binomial factors is the opposite of the answer stated in Example B.
ACTIVITY 7 Continued

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to factoring quadratic expressions.

Answers

5. The length of the large square is $x + 5$, and its width is $x + 3$, so its area is $(x + 5)(x + 3)$. The area of the large square is equal to the sum of the areas of the smaller squares: $x^2 + 5x + 3x + 15 = x^2 + 8x + 15$. The expressions $(x + 5)(x + 3)$ and $x^2 + 8x + 15$ are equivalent, because both are equal to the area of the large square.

6. Both constant terms are negative. Sample explanation: The product of the constant terms is equal to $c$. If $c$ is positive, the constant terms must have the same sign. The sum of the constant terms is equal to $b$, which is negative. Two positive terms cannot have a negative sum, so the constant terms must both be negative.

7. Sample answer: List factor pairs of $-12$, the constant term in the quadratic expression. Then find the sum of the factor pairs. Keep trying different factor pairs until you get a sum of $-4$, the coefficient of the $x$-term in the quadratic expression. The correct factor pair is $2$ and $-6$. Use those numbers as the constant terms in the factored expression: $(x - 2)(x + 6)$. Check your work by multiplying the binomials to see whether you get the original quadratic expression: $(x - 2)(x + 6) = x^2 + 6x - 2x - 12 = x^2 + 4x - 12$.

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand how to factor quadratic expressions whether or not the lead coefficient is equal to one. Make sure students always attempt to find the greatest common factor of a quadratic expression before proceeding. Some students may benefit from making a flow chart or other graphic organizer that describes the process of factoring trinomials.

LESMON 7-2 PRACTICE

Factor each quadratic expression.

8. $2x^2 + 15x + 28$

9. $3x^2 + 25x - 18$

10. $x^2 + x - 30$

11. $x^2 + 15x + 56$

12. $6x^2 - 7x - 5$

13. $12x^2 - 43x + 10$

14. $2x^2 + 5x$

15. $9x^2 - 3x - 2$

16. A customer of Fence Me In wants to increase both the length and width of a rectangular fenced area in her backyard by $x$ feet. The new area in square feet enclosed by the fence is given by the expression $x^2 + 30x + 200$. 

a. Factor the quadratic expression.

b. Reason quantitatively. What were the original length and width of the fenced area? Explain your answer.

Try These B

Factor, and then check by multiplying. Show your work.

a. $10x^2 + 11x + 3$

b. $4x^2 + 17x - 15$

c. $2x^2 - 13x + 21$

d. $6x^2 - 19x - 36$


correct factor pair is $2$ and $-6$, the coefficient of the $x$-term in the quadratic expression. Then find the $c$. If $c$ is positive, the constant terms must have the same sign. The sum of the constant terms is equal to $b$, which is negative. Two positive terms cannot have a negative sum, so the constant terms must both be negative.

Lesson 7-2

Factoring Quadratic Expressions

Check Your Understanding

5. Explain how the graphic organizer shows that $x^2 + 8x + 15$ is equal to $(x + 5)(x + 3)$.

6. Reason abstractly. Given that $b$ is negative and $c$ is positive in the quadratic expression $x^2 + bx + c$, what can you conclude about the signs of the constant terms in the factored form of the expression? Explain your reasoning.

7. Write a set of instructions for a student who is absent, explaining how to factor the quadratic expression $x^2 + 4x - 12$.
Lesson 7-3
Solving Quadratic Equations by Factoring

Learning Targets:
• Solve quadratic equations by factoring.
• Interpret solutions of a quadratic equation.
• Create quadratic equations from solutions.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Paraphrasing, Think-Pair-Share, Create Representations, Quickwrite

To solve a quadratic equation \( ax^2 + bx + c = 0 \) by factoring, the equation must be in factored form to use the Zero Product Property.

You can check your solutions by substituting the values into the original equation.

MATH TIP
The Zero Product Property states that if \( a \cdot b = 0 \), then either \( a = 0 \) or \( b = 0 \).

Example A
Solve \( x^2 + 5x - 14 = 0 \) by factoring.

Original equation \( x^2 + 5x - 14 = 0 \)

Step 1: Factor the left side. \((x + 7)(x - 2) = 0\)

Step 2: Apply the Zero Product Property. \( x + 7 = 0 \) or \( x - 2 = 0 \)

Step 3: Solve each equation for \( x \).

Solution: \( x = -7 \) or \( x = 2 \)

Try These A

a. Solve \( 3x^2 - 17x + 10 = 0 \) and check by substitution.

<table>
<thead>
<tr>
<th>( 3x^2 - 17x + 10 = 0 )</th>
<th>Original equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (3x - 2)(x - 5) = 0 )</td>
<td>Factor the left side.</td>
</tr>
<tr>
<td>( 3x - 2 = 0 ) or ( x - 5 = 0 )</td>
<td>Apply the Zero Product Property.</td>
</tr>
<tr>
<td>( x = \frac{2}{3} ) or ( x = 5 )</td>
<td>Solve each equation for ( x ).</td>
</tr>
</tbody>
</table>

Solve each equation by factoring. Show your work.

b. \( 12x^2 - 7x - 10 = 0 \)  
   \( x = -\frac{2}{3} \) or \( x = \frac{5}{4} \)

c. \( x^2 + 8x - 9 = 0 \)  
   \( x = -9 \) or \( x = 1 \)

d. \( 4x^2 + 12x + 9 = 0 \)  
   \( x = -\frac{3}{2} \)

e. \( 18x^2 - 98 = 0 \)  
   \( x = \frac{7}{3} \) or \( x = -\frac{7}{3} \)

MATH TIP
You can check your solutions by substituting the values into the original equation.

f. \( x^2 + 6x = -8 \)  
   \( x = -2 \) or \( x = -4 \)

g. \( 5x^2 + 2x = 3 \)  
   \( x = -1 \) or \( x = \frac{3}{5} \)

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Lesson 7-3
Activity 7 • Applications of Quadratic Functions 111
In the previous lesson, you were asked to determine the dimensions of a rectangle with an area of 525 ft² that can be enclosed by 100 ft of fencing. You wrote the quadratic equation \( l^2 - 50l + 525 = 0 \) to model this situation, where \( l \) is the length of the rectangle in feet.

1. **a.** Solve the quadratic equation by factoring.
   \( (l - 35)(l - 15) = 0; l = 35 \text{ or } l = 15 \)

   **b.** What do the solutions of the equation represent in this situation?
   *The rectangle could have a length of 35 ft or a length of 15 ft.*

   **c.** What are the dimensions of a rectangle with an area of 525 ft² that can be enclosed by 100 ft of fencing?
   35 ft \( \times \) 15 ft or 15 ft \( \times \) 35 ft

   **d.** Reason quantitatively. Explain why your answer to part c is reasonable. *Sample answer: The area of a rectangle with a length of 35 ft and a width of 15 ft is 525 ft², and the perimeter is 2(35 + 19) = 100 ft. The dimensions give the correct area and perimeter, so the answer is reasonable.*

2. **a.** A park has two rectangular tennis courts side by side. Combined, the courts have a perimeter of 160 yd and an area of 1600 yd².
   \( -l^2 + 80l = 1600 \text{ or equivalent} \)

   **b.** Construct viable arguments. Explain why you need to write the equation in the form \( a^2 + bl + c = 0 \) before you can solve it by factoring. *To solve the equation by factoring, you need to apply the Zero Product Property. You can only apply this property when one side of the equation is equal to 0.*

   **c.** Solve the quadratic equation by factoring, and interpret the solution.
   \( (l - 40)^2 = 0; l = 40; \text{The length of the court is 40 yd.} \)

   **d.** Explain why the quadratic equation has only one distinct solution. *When the equation is factored, both factors are the same, so there is only one value of \( l \) that makes the equation true.*

**ELL Support**

To help struggling students who may have a misconception of the phrase “side by side,” explain to them that the courts are touching, or they are right up against each other. Then demonstrate this with a drawing or by taking two objects (textbooks) and laying them down on a desktop touching against each other, sharing a common edge. If students have the misconception that these courts are not touching, it alters the perimeter, making the problem unsolvable.
Lesson 7-3
Solving Quadratic Equations by Factoring

3. The equation $2x^2 + 9x - 3 = 0$ cannot be solved by factoring. Explain why this is true. The product of 2 and $-3$ is $-6$, and no factor pair of $-6$ has a sum of 9.

Check Your Understanding

4. Explain how to use factoring to solve the equation $2x^2 + 5x = 3$.

5. Critique the reasoning of others. A student incorrectly states that the solution of the equation $x^2 + 2x - 35 = 0$ is $x = -5$ or $x = 7$. Describe the student’s error, and solve the equation correctly.

6. Fence Me In has been asked to install a fence around a cabin. The cabin has a length of 10 yd and a width of 8 yd. There will be a space $x$ yd wide between the cabin and the fence on all sides, as shown in the diagram. The area to be enclosed by the fence is 224 yd$^2$.

   a. Write a quadratic equation that can be used to determine the value of $x$.
   b. Solve the equation by factoring.
   c. Interpret the solutions.

Example B
Write a quadratic equation in standard form with the solutions $x = 4$ and $x = -5$.

Step 1: Write linear equations that correspond to the solutions.

$x - 4 = 0$ or $x + 5 = 0$

Step 2: Write the linear expressions as factors.

$(x - 4)$ and $(x + 5)$

Step 3: Multiply the factors to write the equation in factored form.

$(x - 4)(x + 5) = 0$

Step 4: Multiply the binomials and write the equation in standard form.

$x^2 + x - 20 = 0$

Solution: $x^2 + x - 20 = 0$ is a quadratic equation with solutions $x = 4$ and $x = -5$.

MATH TERMS

The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where $a \neq 0$.

ACTIVITY 7

2-3 (continued) In Item 3, students may attempt to factor the quadratic expression, but they will soon find that it is not possible. It is important for students to realize that not all quadratic trinomials are factorable over the integers.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to solving quadratic equations by factoring. For Item 6, have students explain why one of the solutions needs to be excluded in this situation.

Answers

4. First, set the right side of the equation equal to 0 by subtracting 3 from both sides: $2x^2 + 5x - 3 = 0$. Then factor the quadratic expression on the left side of the equation: $(2x - 1)(x + 3) = 0$. Next, use the Zero Product Property to write two equations: $2x - 1 = 0$ or $x + 3 = 0$. Finally, solve each equation for $x$: $x = \frac{1}{2}$ or $x = -3$.

5. Sample answer: The student factored the equation correctly to get $(x + 7)(x - 5) = 0$, but then used the constant terms of the binomials as the solutions of the equation. Instead, the student should have applied the Zero Product Property to get $x + 7 = 0$ or $x - 5 = 0$. Solving these equations yields the correct solution of $x = -7$ or $x = 5$.

6. a. $4x^2 + 36x + 80 = 224$, or equivalent
   b. $4(x - 3)(x + 12) = 0$; $x = 3$ or $x = -12$
   c. The solution $x = 3$ shows that the space between the cabin and the fence is 3 yd wide. The solution $x = -12$ should be excluded in this situation, because a negative value for the width does not make sense.

Example B Marking the Text,
Paraphrasing, Work Backward,
Debriefing Being able to work from solutions to equations is essential for a complete understanding of solving quadratic equations. Students may need practice writing $x = a$ as a linear equation equal to zero: $x - a = 0$. 

Activity 7 • Applications of Quadratic Functions 113
Example B (continued)  The fact that there are infinitely many equations for a given set of solutions may not be apparent to students. Students have to realize that multiplying both sides of any equation by a real number does not change the solutions. The solution with fraction coefficients is correct, but some students may find it difficult to perform operations with the fractions. By multiplying each equation by the denominator to eliminate the fractions, students will be able to find a quadratic equation with integer coefficients.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to writing equivalent equations for quadratic equations and writing quadratic equations given their solutions.

Answers

7.  $3x^2 - 16x - 12 = 0$
8. The LCD of the fractions in the equation is 6, so multiply both sides of the equation by 6.
   
   $6\left(x^2 - \frac{7}{6}x + \frac{1}{3}\right) = 6(0)$
   
   $6x^2 - 7x + 2 = 0$
9. Yes. The solutions of the quadratic equation $x^2 + 4x + 3 = 0$ are $x = -3$ and $x = -1$. Multiplying both sides of this equation by any real number other than 0 or 1 results in another equation whose solutions are $x = -3$ and $x = -1$.
10. Sample answer: Write a linear equation that corresponds to the solution: $x - 4 = 0$. Write the linear expression as a factor: $(x - 4)$. Multiply the factor by itself to write a quadratic equation in factored form: $(x - 4)(x - 4) = 0$. Then multiply the binomials and write the equation in standard form: $x^2 - 8x + 16 = 0$.

Check Your Understanding

7. Write the equation $3x^2 - 6x = 10x + 12$ in standard form.
8. Explain how you could write the equation $x^2 - \frac{7}{6}x + \frac{1}{3} = 0$ with integer values of the coefficients and constants.
9. Reason quantitatively. Is there more than one quadratic equation whose solutions are $x = -3$ and $x = -1$? Explain.
10. How could you write a quadratic equation in standard form whose only solution is $x = 4$?
Lesson 7-3
Solving Quadratic Equations by Factoring

LESSON 7-3 PRACTICE
Solve each quadratic equation by factoring.
11. $2x^2 - 11x + 5 = 0$
12. $x^2 + 2x = 15$
13. $3x^2 + x - 4 = 0$
14. $6x^2 - 13x - 5 = 0$

Write a quadratic equation in standard form with integer coefficients for which the given numbers are solutions.
15. $x = 2$ and $x = -5$
16. $x = -\frac{2}{3}$ and $x = -5$
17. $x = \frac{3}{5}$ and $x = 3$
18. $x = -\frac{1}{2}$ and $x = \frac{3}{4}$

19. Model with mathematics. The manager of Fence Me In is trying to determine the best selling price for a particular type of gate latch. The function $p(s) = -4s^2 + 400s - 8400$ models the yearly profit the company will make from the latches when the selling price is $s$ dollars.

a. Write a quadratic equation that can be used to determine the selling price that would result in a yearly profit of $1600.

b. Write the quadratic equation in standard form so that the coefficient of $s^2$ is 1.

c. Solve the quadratic equation by factoring, and interpret the solution(s).

d. Explain how you could check your answer to part c.

CONNECT TO ECONOMICS
The selling price of an item has an effect on how many of the items are sold. The number of items that are sold, in turn, has an effect on the amount of profit a company makes by selling the item.
Learning Targets:
- Solve quadratic inequalities.
- Graph the solutions to quadratic inequalities.

SUGGESTED LEARNING STRATEGIES: Identify a Subtask, Guess and Check, Think Aloud, Create Representations, Quickwrite

Factoring is also used to solve quadratic inequalities.

Example A
Solve \( x^2 - x - 6 > 0 \).

Step 1: Factor the quadratic expression on the left \((x + 2)(x - 3) > 0\).

Step 2: Determine where each factor equals zero. \((x + 2) = 0\) at \(x = -2\) \((x - 3) = 0\) at \(x = 3\)

Step 3: Use a number line to visualize the intervals for which each factor is positive \((x + 2)\) and negative \((x - 3)\). (Test a value in each interval to determine the signs.)

\[
\begin{align*}
(x + 2) &> 0 \\
(x - 3) &< 0
\end{align*}
\]

Step 4: Identify the sign of the product of the two factors on each interval.

Step 5: Choose the appropriate interval. Since \(x^2 - x - 6\) is positive \((>)\), the intervals that show \((x + 2)(x - 3)\) as positive represent the solutions.

Solution: \(x < -2\) or \(x > 3\)

Try These A
a. Use the number line provided to solve \(2x^2 + x - 10 \leq 0\).

b. \(x^2 + 3x - 4 < 0\) \(-4 < x < 1\)

c. \(3x^2 + x - 10 \geq 0\) \(x \geq \frac{5}{3}\) or \(x \leq -2\)
Lesson 7-4
More Uses for Factors

A farmer wants to enclose a rectangular pen next to his barn. A wall of the barn will form one side of the pen, and the other three sides will be fenced. He has purchased 100 ft of fencing and has hired Fence Me In to install it so that it encloses an area of at least 1200 ft².

Work with your group on Items 1–5. As you share ideas with your group, be sure to explain your thoughts using precise language and specific details to help group members understand your ideas and your reasoning.

1. **Attend to precision.** If Fence Me In makes the pen 50 ft in length, what will be the width of the pen? What will be its area? Explain your answers.

2. Let \( l \) represent the length in feet of the pen. Write an expression for the width of the pen in terms of \( l \).

3. Write an inequality in terms of \( l \) that represents the possible area of the pen. Explain what each part of your inequality represents.

4. Write the inequality in standard form with integer coefficients.

5. Use factoring to solve the quadratic inequality.

**MATH TIP**

If you multiply or divide both sides of an inequality by a negative number, you must reverse the inequality symbol.
**ACTIVITY 7** Continued

5–7 (continued) Item 6 is just a matter of explaining (in words) what the inequality $40 \leq l \leq 60$ represents. Item 7 involves substituting the extreme values of 40 and 60 into the expression for width in Item 2.

**Check Your Understanding**
Debrief students’ answers to these items to ensure that they understand concepts related to factoring quadratic expressions.

**Answers**

8. a. First determine the value of $x$ for which $(x + 4)$ is equal to 0: $(x + 4) = 0$ when $x = -4$. Then test a value of $x$ less than $-4$ to check whether it makes $(x + 4)$ positive or negative. When $x < -4$, the factor $(x + 4)$ is negative. Next, test a value of $x$ greater than $-4$ to check whether it makes $(x + 4)$ positive or negative. When $x > -4$, the factor $(x + 4)$ is positive. Repeat these steps for the factor $(x - 5)$ to find that it is negative when $x < 5$ and positive when $x > 5$.

b. For intervals on which both factors are positive or both factors are negative, the product $(x + 4)(x - 5)$ is positive. For intervals on which one factor is positive and one factor is negative, the product $(x + 4)(x - 5)$ is negative.

c. The solutions are values of $x$ for which $(x + 4)(x - 5) \geq 0$. So, the solutions are intervals for which the product $(x + 4)(x - 5)$ is positive.

9. The equation $x^2 + 5x - 24 = 0$ has two solutions: $x = -8$ and $x = 3$. The solutions of the inequality $x^2 + 5x - 24 \leq 0$ also include $x = -8$ and $x = 3$ as well as all values of $x$ between $-8$ and $3$. So, the solutions of the inequality are $-8 \leq x \leq 3$.

10. Sample answer: The expression $x^2$ is never negative, so the sum of $x^2$ and 4 is never negative. Because the expression on the left side of the inequality can never be less than 0, the inequality has no real solutions.

**LESSON 7–4 PRACTICE**

<table>
<thead>
<tr>
<th>Solve each inequality.</th>
</tr>
</thead>
<tbody>
<tr>
<td>11. $x^2 + 3x - 10 \geq 0$</td>
</tr>
<tr>
<td>12. $2x^2 + 3x - 9 &lt; 0$</td>
</tr>
<tr>
<td>13. $x^2 + 9x + 18 \leq 0$</td>
</tr>
<tr>
<td>14. $3x^2 - 10x - 8 &gt; 0$</td>
</tr>
<tr>
<td>15. $x^2 - 12x + 27 &lt; 0$</td>
</tr>
<tr>
<td>16. $5x^2 + 12x + 4 &gt; 0$</td>
</tr>
<tr>
<td>17. The function $p(s) = -500s^2 + 15,000s - 100,000$ models the yearly profit Fence Me In will make from installing wooden fences when the installation price is $s$ dollars per foot.</td>
</tr>
<tr>
<td>a. Write a quadratic inequality that can be used to determine the installation prices that will result in a yearly profit of at least $8000.</td>
</tr>
<tr>
<td>b. Write the quadratic inequality in standard form so that the coefficient of $s^2$ is 1.</td>
</tr>
<tr>
<td>c. Make sense of problems. Solve the quadratic inequality by factoring, and interpret the solution(s).</td>
</tr>
</tbody>
</table>

**ASSESS**

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

**ADAPT**

Check students’ answers to the Lesson Practice to ensure that they understand how to use a number line to solve a quadratic inequality. Although some students may be able to determine the solution through visualization of the parabola, it is essential that students master the number line method. This method will be used again when students solve rational and higher order polynomial inequalities.

6. Interpret the solutions of the inequality.
   **The length of the pen must be at least 40 ft and no more than 60 ft.**

7. Use the possible lengths of the pen to determine the possible widths.
   **The width of the pen must be at least 20 ft and no more than 30 ft.**

---

**Check Your Understanding**

8. Consider the inequality $(x + 4)(x - 5) \geq 0$.
   a. Explain how to determine the intervals on a number line for which each of the factors $(x + 4)$ and $(x - 5)$ is positive or negative.
   b. **Reason abstractly.** How do you determine the sign of the product $(x + 4)(x - 5)$ on each interval?
   c. Once you know the sign of the product $(x + 4)(x - 5)$ on each interval, how do you identify the solutions of the inequality?

9. Explain how the solutions of $x^2 + 5x - 24 = 0$ differ from the solutions of $x^2 + 5x - 24 \leq 0$.

10. Explain why the quadratic inequality $x^2 + 4 < 0$ has no real solutions.

---

**LESSON 7–4 PRACTICE**

<table>
<thead>
<tr>
<th>Solve each inequality.</th>
</tr>
</thead>
<tbody>
<tr>
<td>11. $x \leq -5$ or $x \geq 2$</td>
</tr>
<tr>
<td>12. $-3 &lt; x &lt; \frac{3}{2}$</td>
</tr>
<tr>
<td>13. $-6 \leq x \leq -3$</td>
</tr>
<tr>
<td>14. $x &lt; -\frac{2}{3}$ or $x &gt; 4$</td>
</tr>
<tr>
<td>15. $3 &lt; x &lt; 9$</td>
</tr>
<tr>
<td>16. $x &lt; -2$ or $x &gt; -\frac{5}{3}$</td>
</tr>
<tr>
<td>17. a. $-500s^2 + 15,000s - 100,000 \geq 8000$</td>
</tr>
<tr>
<td>b. $s^2 - 30s + 216 \leq 0$</td>
</tr>
<tr>
<td>c. $(s - 12)(s - 18) \leq 0; 12 \leq s \leq 18$ The company will make a profit of at least $8000 when the installation price of the wooden fencing is at least $12 per foot and no more than $18 per foot.</td>
</tr>
</tbody>
</table>
ACTIVITY 7 PRACTICE
Write your answers on notebook paper. Show your work.

Lesson 7-1
A rectangle has perimeter 40 cm. Use this information for Items 1–7.
1. Write the dimensions and areas of three rectangles that fit this description.
2. Let the length of one side be \(x\). Then write a function \(A(x)\) that represents that area of the rectangle.
3. Graph the function \(A(x)\) on a graphing calculator. Then sketch the graph on grid paper, labeling the axes and using an appropriate scale.
4. An area of 96 \(\text{cm}^2\) is possible. Use \(A(x)\) to demonstrate this fact algebraically and graphically.
5. An area of 120 \(\text{cm}^2\) is not possible. Use \(A(x)\) to demonstrate this fact algebraically and graphically.
6. What are the reasonable domain and reasonable range of \(A(x)\)? Express your answers as inequalities, in interval notation, and in set notation.
7. What is the greatest area that the rectangle could have? Explain.

Use the quadratic function \(f(x) = x^2 - 6x + 8\) for Items 8–11.
8. Graph the function.
9. Write the domain and range of the function as inequalities, in interval notation, and in set notation.
10. What is the function’s \(y\)-intercept?
   A. 0  B. 2  C. 4  D. 8
11. Explain how you could use the graph of the function to solve the equation \(x^2 - 6x + 8 = 3\).

12. Factor \(x^2 + 11x + 28\) by copying and completing the graphic organizer. Then check by multiplying.

13. Factor each quadratic expression.
   a. \(2x^2 - 3x - 27\)  b. \(4x^2 - 121\)  c. \(6x^2 + 11x - 10\)  d. \(3x^2 + 7x + 3\)  e. \(5x^2 - 42x - 27\)  f. \(4x^4 - 4x - 35\)  g. \(36x^2 - 100\)  h. \(12x^3 + 60x + 75\)

14. Given that \(b\) is positive and \(c\) is negative in the quadratic expression \(x^2 + bx + c\), what can you conclude about the signs of the constant terms in the factored form of the expression? Explain your reasoning.

15. The area in square inches of a framed photograph is represented graphically by there being no point of intersection between \(A(x)\) and \(A(x)\). Explain your answer.

ACTIVITY 7 Continued

ACTIVITY PRACTICE

1. Sample answers: 5 \(\text{cm}\) \(\times\) 15 \(\text{cm}\), area = 75 \(\text{cm}^2\); 9 \(\text{cm}\) \(\times\) 11 \(\text{cm}\), area = 99 \(\text{cm}^2\); 2 \(\text{cm}\) \(\times\) 18 \(\text{cm}\), area = 36 \(\text{cm}^2\)
2. \(A(x) = (20 - x)x = 20x - x^2\)
3. \(A(x)\)

4. \(20x - x^2 = 96\) has solutions of \(x = 8\) and \(x = 12\) represented graphically by the points of intersection of \(A(x) = 96\) and \(A(x)\).
5. The equation \(20x - x^2 = 120\) has no real-number solutions, which is represented graphically by there being no point of intersection between \(A(x)\) and \(A(x)\).
6. Domain: \(0 < x < 20\), \(0 < x < 20\); range: \(0 < x < 20\), \(0 < x < 20\), \(0 < A < 100\), \(0, 100\), \(0 < A < 100\)
7. 100 \(\text{cm}^2\); Sample explanation: The graph shows that the maximum value of the function occurs when \(x = 10\), and \(A(10) = 20(10) - 10^2 = 100\).
8. \(f(x)\)

9. Domain: \(-\infty < x < \infty\), \(-\infty, \infty\), \(\{x \in R, 0 < x < 20\}; range: y \geq 1 [-1, \infty), \{y \in R, y \geq -1\}
10. D
11. Find the points on the graph of \(f(x)\) where \(f(x) = 3\). The \(x\)-coordinates of these points are the solutions of \(x^2 - 6x + 8 = 3\). Because \(f(x) = 3\) when \(x = 1\) and when \(x = 5\), the solutions of \(x^2 - 6x + 8 = 3\) are \(x = 1\) and \(x = 5\).
12. \(x^2 + 11x + 28 = (x + 4)(x + 7)\)
16. a. \[ x = -\frac{3}{2}; x = 4 \]
   b. \[ x = \frac{1}{3}; x = -2 \]
   c. \[ x = \frac{3}{2} \]
   d. \[ x = \frac{2}{3}; x = -\frac{2}{3} \]
   e. \[ x = \frac{4}{3}; x = -\frac{1}{2} \]

17. More than one correct equation is possible; other correct equations would be real-number multiples of the equations given.
   a. \[ x^2 + 3x - 40 = 0 \]
   b. \[ 3x^2 - 14x + 8 = 0 \]
   c. \[ 10x^2 + 9x - 7 = 0 \]
   d. \[ x^2 - 12x + 36 = 0 \]

18. No. Sample explanation: The student is assuming that if a product is equal to 2, then one of the factors must be equal to 2. This assumption is incorrect. For example, the product \(4 \times \frac{1}{2}\) is equal to 2, but neither of the factors is equal to 2.

19. \( B \)

20. \[ b^2 + 30b - 5400 = 0 \]

21. \((b + 90)(b - 60) = 0; b = -90\) or \(b = 60\). The solution \(b = -90\) must be excluded, because \(b\) represents the base of a triangle, and it does not make sense for the base to be negative. The solution \(b = 60\) shows that the base of the triangle measures 60 ft.

22. \( x < -4 \) or \( x > 6 \); Sample explanation: The factor \((x + 4)\) is negative for \(x < -4\) and positive for \(x > 4\). The factor \((x - 6)\) is negative for \(x < 6\) and positive for \(x > 6\). Both factors are negative, which means their product is positive when \(x < -4\); and both factors are positive, which also means their product is positive when \(x > 6\).

23. a. \( -1 \leq x \leq 4 \)
   b. \( x < -\frac{2}{3} \) or \( x > 3 \)
   c. no real solutions
   d. \( x \leq -3 \) or \( x \geq -1 \)
   e. \(-3 \leq x \leq 7 \)
   f. \(-\frac{2}{3} < x < 3 \)

24. \( -16t^2 + 20t + 6 \geq 10 \)

25. \( 16t^2 - 20t + 4 \leq 0 \)

26. \( t \leq \frac{1}{4} \); The ball is at least 10 ft above the ground between \( \frac{1}{4} \) second and 1 second after it is thrown.

### ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.
### Introduction to Complex Numbers

**Cardano’s Imaginary Numbers**

**Lesson 8-1 The Imaginary Unit, i**

**Learning Targets:**
- Know the definition of the complex number $i$.
- Know that complex numbers can be written as $a + bi$, where $a$ and $b$ are real numbers.
- Graph complex numbers on the complex plane.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Interactive Word Wall, Marking the Text, Think-Pair-Share, Quickwrite

The equation $x^2 + 1 = 0$ has special historical and mathematical significance. At the beginning of the sixteenth century, mathematicians believed that the equation had no solutions.

1. **Why would mathematicians of the early sixteenth century think that $x^2 + 1 = 0$ had no solutions?**
   - If $x^2 + 1 = 0$, then $x^2 = -1$, and the square root of a negative number is not a real number.

A breakthrough occurred in 1545 when the talented Italian mathematician Girolamo Cardano (1501–1576) published his book, *Ars Magna* (The Great Art). In the process of solving one cubic (third-degree) equation, he encountered—and was required to make use of—the square roots of negative numbers. While skeptical of their existence, he demonstrated the situation encountered—and was required to make use of—the square roots of negative numbers. He then added, subtract, and multiply complex numbers. They factor quadratic expressions with complex conjugates.

Finally, students solve equations with complex solutions. Throughout this activity, make connections between properties of real numbers and properties of complex numbers.

### Lesson 8-1

#### PLAN

**Pacing:** 1 class period

** Chunking the Lesson**

#1  #2–4  
#5–6 Example A

Check Your Understanding

#11

Check Your Understanding

Example B

Check Your Understanding

Lesson Practice

#### TEACH

**Bell-Ringer Activity**

Have students solve the following equations containing square roots.

1. $\sqrt{x} = 4$  \[x = 16\]
2. $\sqrt{x} + 2 = 7$  \[x = 47\]
3. $x - 1 = \sqrt{4x} - 8$  \[x = 3\]

Discuss the methods used to solve each equation.

**Activating Prior Knowledge**

This item provides a good opportunity for formative assessment regarding solving quadratic equations of the form $x^2 + a = 0$.

**2–4 Guess and Check, Create Representations, Activating Prior Knowledge, Debriefing**

Most students will likely be able to answer Item 2 using a guess-and-check strategy. For Items 3 and 4, opportunities exist for formative assessment of students’ abilities to create a quadratic equation from a verbal description and for them to solve the equation using factoring and the Quadratic Formula.

### Common Core State Standards for Activity 8

- **HN-CN.A.1** Know there is a complex number $i$ such that $i^2 = -1$, and every complex number has the form $a + bi$ with $a$ and $b$ real.
- **HN-CN.A.2** Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
- **HN-CN.A.3(+)** Find the conjugate of a complex number; use conjugates to find moduli and quotients of complex numbers.
- **HN-CN.B.4(+)** Represent complex numbers on the complex plane in rectangular and polar form (including real and imaginary numbers), and explain why the rectangular and polar forms of a given complex number represent the same number.

[Note: HSN-CN.A.3 and HSN-CN.B.4 are introduced in this activity but are also addressed in higher level mathematics courses.]
5–6 Create Representations, Activating Prior Knowledge

After debriefing Items 3 and 4, students should be able to write the equation in Item 5 more easily. In Item 6, students should recognize the equation may be solved using the Quadratic Formula, but not by factoring.

Developing Math Language

In mathematics, imaginary numbers are not "make-believe"; they are a set of numbers that do exist. Imaginary numbers exist so that negative numbers can have square roots and certain equations can have solutions. Additionally, imaginary numbers have significant technological applications, particularly in the fields of electronics and engineering.

Universal Access

A misconception that some students have is to not realize that \(\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}\) is true for nonnegative numbers only. For example, students may incorrectly solve this problem as follows:

\[
-\sqrt{3} \cdot \sqrt{-11} = \sqrt{-3 \cdot -11} = \sqrt{33}
\]

However, this must be solved by rewriting the radicals using \(i\):

\[
-\sqrt{3} \cdot \sqrt{-11} = -i\sqrt{3} \cdot i\sqrt{11} = -i^2\sqrt{3\cdot11} = \sqrt{33}
\]

MATH TERMS

An imaginary number is any number of the form \(bi\), where \(b\) is a real number and \(i = \sqrt{-1}\).

MATH TIP

You can solve a quadratic equation by graphing, by factoring, or by using the Quadratic Formula, \(x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\). You can use it to solve quadratic equations in the form \(ax^2 + bx + c = 0\), where \(a \neq 0\).

CONNECT TO HISTORY

The Ars Magna was first published under the title Artis Magnae, Sive de Regulis Algebraicis Liber Unus (Book Number One about The Great Art, or The Rules of Algebra). The main focus of this work involved methods of solving third- and fourth-degree equations. The "great art" of algebra described by Cardano was in comparison to the "lesser art" of the day, arithmetic.

Rafael Bombelli (1526–1572), an Italian architect and engineer, was intrigued by Cardano's methods and formalized the rules for operations with complex numbers.

Common Core State Standards for Activity 8 (continued)

- HSN-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.
- HSN-CN.C.8 Extend polynomial identities to the complex numbers.

Lesson 8-1

The Imaginary Unit, \(i\)

4. Solve your equation in Item 3 in two different ways. Explain each method.

\[x(10 - x) = 21, \text{ so } x^2 - 10x + 21 = 0.\]

By factoring, \((x - 3)(x - 7) = 0\), so \(x = 3\) or \(x = 7\).

Using the Quadratic Formula on \(x^2 - 10x + 21 = 0\) yields

\[x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(21)}}{2} = \frac{10 \pm \sqrt{100 - 84}}{2} = \frac{10 \pm 6}{2} = 5 \pm 2, \text{ so } x = 3 \text{ or } x = 7.\]

5. Write an equation that represents the problem that Cardano posed.

\[x(10 - x) = 40\]

6. Cardano claimed that the solutions to the problem are \(x = 5 + \sqrt{-15}\) and \(x = 5 - \sqrt{-15}\). Verify his solutions by using the Quadratic Formula with the equation in Item 5.

\[x = \frac{10 \pm \sqrt{100 - 4(1)(40)}}{2} = \frac{10 \pm \sqrt{-60}}{2} = \frac{10 \pm 2\sqrt{-15}}{2} = 5 \pm \sqrt{-15}\]

Cardano avoided any more problems in Ars Magna involving the square root of a negative number. However, he did demonstrate an understanding about the properties of such numbers. Solving the equation \(x^2 + 1 = 0\) yields the solutions \(x = \sqrt{-1}\) and \(x = -\sqrt{-1}\). The number \(\sqrt{-1}\) is represented by the symbol \(i\), the imaginary unit. You can say \(i = \sqrt{-1}\). The imaginary unit \(i\) is considered the solution to the equation \(x^2 + 1 = 0\), or \(x^2 = -1\).

To simplify an imaginary number \(\sqrt{-s}\), where \(s\) is a positive number, you can write \(\sqrt{-s} = i\sqrt{s}\).
Lesson 8-1
The Imaginary Unit, $i$

**Example A**

Write the numbers $\sqrt{-17}$ and $\sqrt{-9}$ in terms of $i$.

**Step 1:** Definition of $\sqrt{-5}$

$$\sqrt{-17} = -i \sqrt{17}$$

**Step 2:** Take the square root of 9.

$$\sqrt{-9} = i \sqrt{9}$$

Solution: $\sqrt{-17} = -i \sqrt{17}$ and $\sqrt{-9} = 3i$

**Try These A**

Write each number in terms of $i$.

- a. $\sqrt{-25}$ $5i$
- b. $\sqrt{-7}$ $i \sqrt{7}$
- c. $\sqrt{-12}$ $2i \sqrt{3}$
- d. $\sqrt{-150}$ $5i \sqrt{6}$

Check Your Understanding

7. Make use of structure. Rewrite the imaginary number $4i$ as the square root of a negative number. Explain how you determined your answer.

8. Simplify each of these expressions: $-\sqrt{20}$ and $\sqrt{-20}$. Are the expressions equivalent? Explain.

9. Write each number in terms of $i$.

   - a. $\sqrt{-98}$
   - b. $-\sqrt{-27}$
   - c. $\sqrt{(-8)(3)}$
   - d. $\sqrt{25 - 4(2)(6)}$

10. Why do you think imaginary numbers are useful for mathematicians?

11. Write the solutions to Cardano’s problem, $x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$, using the imaginary unit $i$.

   - $5 + 2i \sqrt{15}$ and $5 - 2i \sqrt{15}$

ACTIVITY 8 Continued

**Example A** Discussion Groups, Activating Prior Knowledge, Debriefing

The concept of the imaginary unit will be new, and somewhat confusing, to many students. Tell students that complex numbers, although presented in a theoretical way, play an important part in advanced studies of applied sciences like physics and electrical engineering.

**Check Your Understanding**

Debrief students’ answers to these items to ensure that they understand concepts related to basic concepts of imaginary numbers.

**Answers**

7. $\sqrt{-16}$. Sample explanation: First write $4i$ as a square root: $4i = \sqrt{(4i)^2}$. Apply the Power of a Product Property: $\sqrt{(4i)^2} = \sqrt{4^2 \cdot i^2}$. $i^2$ is equal to $-1$, so $\sqrt{4^2 \cdot (-1)} = \sqrt{16 \cdot (-1)} = \sqrt{-16}$.

8. $-\sqrt{20} = -2 \sqrt{5}$ and $\sqrt{-20} = 2i \sqrt{5}$; The expressions are not equivalent. Sample explanation: $-\sqrt{20}$ represents the opposite of the square root of a positive number. The square root of a positive number is a real number, so its opposite is also a real number. $\sqrt{-20}$ represents the square root of a negative number. The square root of any negative number is an imaginary number.

9. a. $7i \sqrt{2}$
   - b. $-3i \sqrt{5}$
   - c. $2i \sqrt{6}$
   - d. $i \sqrt{23}$

10. Sample answer: Imaginary numbers can be used when solving quadratic equations that do not have real solutions.

**11 Activating Prior Knowledge, Think-Pair-Share**

Have students work in pairs to apply what they have learned so far, by going back to Item 6. Cardano’s solutions to a problem where two numbers have a sum of 10 and a product of 40 are:

$x = 5 + \sqrt{-15}$ and $x = 5 - \sqrt{-15}$.

These can be rewritten as follows:

$x = 5 + \sqrt{-1} \cdot \sqrt{15}$ and $x = 5 - \sqrt{-1} \cdot \sqrt{15}$

$x = 5 + i \sqrt{15}$ and $x = 5 - i \sqrt{15}$

Ask a student volunteer to present these to the class.
The set of complex numbers consists of the real numbers and the imaginary numbers. A complex number has two parts: the real part $a$ and the imaginary part $bi$. For example, in $2 + 3i$, the real part is 2 and the imaginary part is $3i$.

Complex numbers in the form $a + bi$ can be represented geometrically as points in the complex plane. The complex plane is a rectangular grid, similar to the Cartesian plane, with the horizontal axis representing the real part $a$ of a complex number and the vertical axis representing the imaginary part $bi$ of a complex number. The point $(a, b)$ on the complex plane represents the complex number $a + bi$.

Example B

Point $A$ represents $0 + 4i$.

Point $B$ represents $-3 + 2i$.

Point $C$ represents $1 - 4i$.

Point $D$ represents $3 + 0i$.

Try These B

a. Graph $2 + 3i$ and $-3 - 4i$ on the complex plane above.

Graph each complex number on a complex plane grid.

b. $2 + 5i$

c. $4 - 3i$

d. $-1 + 3i$

e. $-2i$

f. $-5$
Lesson 8-1
The Imaginary Unit, i

Check Your Understanding

15. Reason abstractly. Compare and contrast the Cartesian plane with the complex plane.
16. What set of numbers do the points on the real axis of the complex plane represent? Explain.
17. Name the complex number represented by each labeled point on the complex plane below.

LESSON 8-1 PRACTICE

18. Write each expression in terms of i.
   a. \( \sqrt{-49} \)
   b. \( \sqrt{-13} \)
   c. \( 3 + \sqrt{-8} \)
   d. \( 5 - \sqrt{-36} \)

19. Identify the real part and the imaginary part of the complex number \( 16 - i\sqrt{6} \).


21. Draw the complex plane. Then graph each complex number on the plane.
   a. \( 6i \)
   b. \( 3 + 4i \)
   c. \( -2 - 5i \)
   d. \( 4 - i \)
   e. \( -3 + 2i \)

22. The sum of two numbers is 8, and their product is 80.
   a. Let \( x \) represent one of the numbers, and write an expression for the other number in terms of \( x \). Use the expressions to write an equation that models the situation given above.
   b. Use the Quadratic Formula to solve the equation. Write the solutions in terms of \( i \).

MATH TIP
\( \pi \) is the ratio of a circle's circumference to its diameter. \( \pi \) is an irrational number, and its decimal form neither terminates nor repeats.

LESSON 8-1 PRACTICE

18. a. \( 7i \)
   b. \( i\sqrt{13} \)
   c. \( 3 + 2i\sqrt{2} \)
   d. \( 5 - 6i \)

19. real part: 16; imaginary part: \( -\sqrt{6} \)

20. Yes. \( \pi \) is an irrational number, and all irrational numbers are real numbers. Because \( \pi \) is a real number, it is also a complex number that can be written as \( \pi + 0i \).

22. a. \( 8 - x; x(8 - x) = 80 \)
   b. \( x = 4 + 8i \) or \( x = 4 - 8i \)

ACTIVITY 8

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to graphing complex numbers on the complex plane.

Answers

15. Sample answer: Both are formed by the intersection of a horizontal axis and a vertical axis. In the Cartesian plane, both axes (the \( x \)-axis and the \( y \)-axis) represent real numbers. In the complex plane, the horizontal, or real, axis represents the real numbers, and the vertical, or imaginary, axis represents the imaginary numbers. On the Cartesian plane, an ordered pair \( (x, y) \) gives the location of a point that is a horizontal distance of \( x \) units from the origin and a vertical distance of \( y \) units from the origin. On the complex plane, an ordered pair \( (a, b) \) represents the location of the complex number \( a + bi \).

16. The set of real numbers; Points on the real axis represent complex numbers with an imaginary part that is equal to 0. In other words, the numbers have the form \( a + 0i \), where \( a \) is a real number.

17. A: \( -4 + 4i \)
   B: \( 3i \)
   C: \( 5 + i \)
   D: \( 2 - 4i \)
   E: \( -3 - 5i \)

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.
See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students' answers to the Lesson Practice to ensure that they understand both the symbolic and the graphical representations of complex numbers. Students should also be comfortable solving a quadratic equation that has complex solutions. Some students may benefit from a Venn diagram depicting the set of complex numbers and all of its subsets. Students can write examples of each type of number within the Venn diagram as a reference.
Lesson 8-2

Learning Targets:
- Add and subtract complex numbers.
- Multiply and divide complex numbers.

**SUGGESTED LEARNING STRATEGIES:** Group Presentation, Self Revision/Peer Revision, Look for a Pattern, Quickwrite

Perform addition of complex numbers as you would for addition of binomials of the form $a + bx$. To add such binomials, you collect like terms.

**Example A**

Addition of Binomials

<table>
<thead>
<tr>
<th>Step 1</th>
<th>Collect like terms.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(5 + 4x) + (-2 + 3x)$</td>
<td>$= (5 - 2) + (4x + 3x)$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Step 2</th>
<th>Simplify.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= 3 + 7x$</td>
<td>$= 3 + 7i$</td>
</tr>
</tbody>
</table>

**Try These A**

Add the complex numbers.

a. $(6 + 5i) + (4 - 7i)$ $10 - 2i$

b. $(-5 + 3i) + (-3 - i)$ $-8 + 2i$

c. $(2 + 3i) + (-2 - 3i)$ $0$

1. Express regularity in repeated reasoning. Use Example A above and your knowledge of operations of real numbers to write general formulas for the sum and difference of two complex numbers.

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$
Lesson 8-2
Operations with Complex Numbers

2. Find each sum or difference of the complex numbers.
   a. $(12 - 13i) - (-5 + 4i)$  \[ 17 - 17i \]
   b. $\left( \frac{1}{2} - i \right) + \left( \frac{5}{2} + 9i \right)$  \[ 3 + 8i \]
   c. $(\sqrt{2} - 7i) + (2 + i\sqrt{3})$  \[ (2 + \sqrt{2}) + (\sqrt{3} - 7)i \]
   d. $(8 - 5i) - (3 + 5i) + (-5 + 10i)$  \[ 0 + 10i = 0 \]

Check Your Understanding

3. Recall that the sum of a number and its additive inverse is equal to 0. What is the additive inverse of the complex number $3 - 5i$? Explain how you determined your answer.
4. Reason abstractly. Is addition of complex numbers commutative? In other words, is $(a + bi) + (c + di)$ equal to $(c + di) + (a + bi)$? Explain your reasoning.
5. Give an example of a complex number you could subtract from $8 + 3i$ that would result in a real number. Show that the difference of the complex numbers is equal to a real number.

Perform multiplication of complex numbers as you would for multiplication of binomials of the form $ax + bx$. The only change in procedure is to substitute $i^2$ with $-1$.

Example B
Multiply Binomials
$(2 + 3x)(4 - 5x)$
$2(4) + 2(-5x) + 3x(4) + 3x(-5x)$
$8 - 10x + 12x - 15x^2$
$8 + 2x - 15x^2$

Multiply Complex Numbers
$(2 + 3i)(4 - 5i)$
$2(4) + 2(-5i) + 3i(4) + 3i(-5i)$
$8 - 10i + 12i - 15i^2$
$8 + 2i - 15(-1)$
$= 23 + 2i$

Try These B
Multiply the complex numbers.
   a. $(6 + 5i)(4 - 7i)$  \[ 59 - 22i \]
   b. $(2 - 3i)(3 - 2i)$  \[ 0 - 13i = -13i \]
   c. $(5 + i)(5 + i)$  \[ 24 + 10i \]
ACTIVITY 8

6–7 Activating Prior Knowledge, Look for a Pattern, Group
Presentation, Debriefing Students will again use properties of real numbers to generalize the rules for multiplication of complex numbers. Make certain that students acknowledge the use of these properties in their discussions and explanations. Additionally, the fact that \( i^2 = -1 \) and the pattern that evolves with powers of \( i \) in the accompanying Math Tip should be highlighted and explored with at least a few additional examples.

**Check Your Understanding**

Debrief students’ answers to these items to ensure that they understand concepts related to multiplying complex numbers.

**Answers**

8. a. 34
   b. 52
   c. 65
9. Sample answer: Each problem has the form \((a + bi)(a - bi)\). All of the products are real numbers.
10. Sample answer: For \((a + bi)(c + di)\), the imaginary terms of the product are \((ad)\) and \((bc)i\). If these imaginary terms are opposites, their sum will be 0 or 0, leaving only the real terms of the product.
11. Yes. Sample explanation: Let \( b \) and \( c \) be any 2 imaginary numbers, where \( b \) and \( c \) are real. Find their product: \((bi) \cdot (ci) = (bc)i^2 = (bc)(-1) = -bc\). Because \( b \) and \( c \) are real numbers, the product \(-bc\) is a real number.

**MATH TIP**

Since \( i = \sqrt{-1} \), the powers of \( i \) can be evaluated as follows:

\[
\begin{align*}
i^1 &= i \\
i^2 &= -1 \\
i^3 &= i^2 \cdot i = -1 \cdot i = -i \\
i^4 &= i^3 \cdot i = -i \cdot i = 1
\end{align*}
\]

Since \( i^4 = 1 \), further powers repeat the pattern shown above.

\[
\begin{align*}
i^5 &= i^4 \cdot i = i \\
i^6 &= i^5 \cdot i = i^2 = -1 \\
i^7 &= i^6 \cdot i = -i \\
i^8 &= i^7 \cdot i = i^2 = 1
\end{align*}
\]

**Mini-Lesson: Powers of \( i \)**

If students need additional help with writing expressions as a power of \( i \) with an exponent between 1 and 4, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.

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**Lesson 8-2**

**Operations with Complex Numbers**

6. **Express regularity in repeated reasoning.** Use Example B and your knowledge of operations of real numbers to write a general formula for the multiplication of two complex numbers.

\[
(a + bi) \cdot (c + di) = (ac - bd) + (ad + bc)i
\]

7. Use operations of complex numbers to verify that the two solutions that Cardano found, \( x = 5 + \sqrt{-15} \) and \( x = 5 - \sqrt{-15} \), have a sum of 10 and a product of 40.

\[
\begin{align*}
(5 + \sqrt{15}) + (5 - \sqrt{15}) &= 10 \\
(5 + \sqrt{15})(5 - \sqrt{15}) &= 25 - 5i\sqrt{15} + 5i\sqrt{15} - 15^2 = 25 + 15 = 40
\end{align*}
\]

**Check Your Understanding**

8. Find each product.
   a. \((5 + 3i)(5 - 3i)\)
   b. \((-6 - 4i)(-6 + 4i)\)
   c. \((8 + i)(8 - i)\)

9. What pattern do you observe in the products in Item 8?

10. Explain how the product of two complex numbers can be a real number, even though both factors are not real numbers.

11. **Critique the reasoning of others.** A student claims that the product of any two imaginary numbers is a real number. Is the student correct? Explain your reasoning.
Lesson 8-2
Operations with Complex Numbers

The complex conjugate of \(a + bi\) is defined as \(a - bi\). For example, the complex conjugate of \(2 + 3i\) is \(2 - 3i\).

12. A special property of multiplication of complex numbers occurs when a number is multiplied by its conjugate. Multiply each number by its conjugate and then describe the product when a number is multiplied by its conjugate.

\(a\). \(2 - 9i\) \((2 - 9i)(2 + 9i) = 4 - 81i^2 = 4 + 81 = 85\)

\(b\). \(-5 + 2i\) \((-5 + 2i)(-5 - 2i) = 25 - 4i^2 = 29\)

Sample answer: A complex number multiplied by its conjugate results in a real number. The product is the sum of the squares of \(a\) and \(b\).

13. Write an expression to complete the general formula for the product of a complex number and its complex conjugate.

\((a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2i^2 = a^2 + b^2\)

To divide two complex numbers, start by multiplying both the dividend and the divisor by the conjugate of the divisor. This step results in a divisor that is a real number.

Example C
Divide \(\frac{4 - 5i}{2 + 3i}\).

Step 1: Multiply the numerator and denominator by the complex conjugate of the divisor.

\[
\frac{4 - 5i}{2 + 3i} \cdot \frac{2 - 3i}{2 - 3i} = \frac{(2 - 3i)(2 - 3i)}{2^2 - (3i)^2} = \frac{4 - 22i + 15i^2}{4 - 9i^2} = \frac{4 - 22i - 15}{4 + 9}
\]

Step 2: Simplify and substitute \(-i\) for \(i^2\).

\[
\frac{4 - 22i - 15}{4 + 9} = \frac{-11 - 22i}{13} = -\frac{7}{13} + \frac{22}{13}i
\]

Step 3: Simplify and write in the form \(a + bi\).

\[
\frac{4 - 5i}{2 + 3i} = -\frac{7}{13} + \frac{22}{13}i
\]

Solution: \(\frac{4 - 5i}{2 + 3i} = -\frac{7}{13} + \frac{22}{13}i\)

ACTIVITY 8

Developing Math Language
The complex conjugate of a complex number \(a + bi\) is \(a - bi\). What is special about the relationship between a complex number and its conjugate is that their product is always a real number. In mathematics, a conjugate is a binomial formed from another binomial by changing the sign of the second term. The other place students have seen a conjugate is when rationalizing the denominator containing a radical. For example, \(\frac{1}{1 + \sqrt{2}}\) is rationalized by multiplying both the numerator and the denominator by the conjugate of the denominator:

\[
\frac{1}{1 + \sqrt{2}} \cdot \frac{1 - \sqrt{2}}{1 - \sqrt{2}} = \frac{1 - \sqrt{2}}{1 - 2} = -1 + \sqrt{2}.
\]

So, just as multiplying a binomial with a radical by its conjugate “clears” the radical sign, multiplying a binomial with an imaginary number “clears” the imaginary number.

12–13 Activating Prior Knowledge

While the process of finding the conjugate of a complex number is relatively simple, many students confuse this with finding the opposite of a complex number by changing the signs of both the real and imaginary parts. Make sure students realize that the product of a complex number and its conjugate will always be a real number, generalizing that result in Item 13.

Example C Quickwrite, Activating Prior Knowledge

Have students summarize the process of dividing complex numbers and explain why you can perform this process using the complex conjugate. Ask them to write down anything they have already learned in algebra that this process reminds them of.

Some may articulate that you can multiply by the complex conjugate as a way to get the quotient in standard form of a complex number, \(a + bi\). Make sure students understand that multiplying both the numerator and denominator by the complex conjugate of the denominator does not change the “value” because you are multiplying by an expression that is equal to 1. This process has some similarities to rationalizing the denominator.
ACTIVITY 8 Continued

Example C (continued) Two different yet appropriate answers exist for Try These C Item a, and both should be explored regardless of student response. The identity of multiplication (because the number is essentially multiplied by 1) and inverse operations (multiply the quotient by the denominator for a product equal to the numerator) are concepts that warrant exploration.

Technology Tip

Students should learn to use their calculators to perform operations using $i$. This will give them another tool to check their work. Help them find $i$ on their calculator — on TI graphing calculators, press \[2ND\] \[\sqrt{}\]. Remind students to be careful to close parentheses when entering operations.

14 Group Presentation, Debriefing

Students generalize the operation of division of complex numbers. While this may appear to be a more daunting task than previous generalizations of operations, students will be successful, especially if they recognize that the denominator can be found easily using Item 13.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to dividing complex numbers.

Answers

15. The quotient of 2 imaginary numbers is a real number. Sample example:

\[
\frac{3i}{4i} = \frac{3}{4} \cdot \frac{4i}{4i} = \frac{12i^2}{16} = \frac{12}{16} = \frac{3}{4}
\]

16. The quotient of a real number and an imaginary number is an imaginary number. Sample example:

\[
\frac{3}{4i} = \frac{3}{4i} \cdot \frac{-4i}{-4i} = \frac{-12i^2}{16i} = \frac{-12}{16} = -\frac{3}{4}i
\]

17. D

18. Sample explanation: I know that $i^{-1}$ is equal to \(\frac{1}{i}\). To divide 1 by $i$, multiply the dividend and the divisor by the conjugate of $i$:

\[
\frac{1}{i} \cdot \frac{-i}{-i} = \frac{-1}{-i} = \frac{1}{i} = -i
\]

Try These C

a. In Example C, why is the quotient \(\frac{7}{13} - \frac{22}{13}i\) equivalent to the original expression \(\frac{4 - 5i}{2 + 3i}\)? There are two acceptable responses.

First, multiplication by \(\frac{2}{2} - \frac{3}{2}i\) is the equivalent of multiplication by 1, the multiplicative identity. Therefore the result is equivalent to the original expression.

Also, multiplying \(\frac{7}{13} - \frac{22}{13}i\) by the divisor, \(2 + 3i\), yields \(4 - 5i\), as division is the inverse operation of multiplication.

\[
\left(\frac{-7}{13} \cdot \frac{22}{13}i\right)(2 + 3i) = -\frac{14}{13} \cdot \frac{21}{13}i - \frac{44}{13}i - \frac{66}{13}i^2 = \left(\frac{66}{13} \cdot \frac{14}{13}i\right) + \left(\frac{-21}{13} \cdot \frac{44}{13}i\right)
\]

\[
\frac{52}{13} - \frac{66}{13}i = 4 - 5i
\]

Divide the complex numbers. Write your answers on notebook paper. Show your work.

b. \(\frac{5 - 2i}{2 + 3i}\) c. \(\frac{7 + 26i}{25 + 26i}\) d. \(\frac{-1 - i}{\sqrt{3} + 4i}\)

主要原因：2

15. Express regularity in repeated reasoning. Use Example C and your knowledge of operations of real numbers to write a general formula for the division of two complex numbers.

\[
(a + bi) \div (c + di) = \frac{(ac + bd)i}{c^2 + d^2}
\]

Check Your Understanding

15. Make a conjecture about the quotient of two imaginary numbers where the divisor is not equal to 0i. Is the quotient real, imaginary, or neither? Give an example to support your conjecture.

16. Make a conjecture about the quotient of a real number divided by an imaginary number not equal to 0i. Is the quotient real, imaginary, or neither? Give an example to support your conjecture.

17. Which of the following is equal to \(i^{-1}\)?

A. 1 B. \(-1\) C. \(i\) D. \(-i\)

18. Explain your reasoning for choosing your answer to Item 17.
Lesson 8-2
Operations with Complex Numbers

LESSON 8-2 PRACTICE
19. Find each sum or difference.
   a. \((6 - 5i) + (-2 + 6i)\)
   b. \((4 + i) + (-4 + i)\)
   c. \((5 - 3i) - (3 - 5i)\)
   d. \((-3 + 8i) - \left(\frac{3}{2} + \frac{1}{2}i\right)\)
20. Multiply. Write each product in the form \(a + bi\).
   a. \((2 + 3i)(3 - i)\)
   b. \((-5 + 8i)(2 - i)\)
   c. \((8 + 15i)(8 - 15i)\)
   d. \((8 - 4i)(5i)\)
21. Divide. Write each quotient in the form \(a + bi\).
   a. \(\frac{1 + 4i}{4 - i}\)
   b. \(-\frac{2 + 5i}{3 - 4i}\)
   c. \(\frac{7 - 3i}{i}\)
22. Use substitution to show that the solutions of the equation \(x^2 - 4x + 20 = 0\) are \(x = 2 + 4i\) and \(x = 2 - 4i\).
23. Make use of structure. What is the sum of any complex number \(a + bi\) and its complex conjugate?
24. Explain how to use the Commutative, Associative, and Distributive Properties to add the complex numbers \(5 + 8i\) and \(6 + 2i\).

ACTIVITY 8
continued

My Notes

ASSESS

Students' answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 8-2 PRACTICE
19. a. \(4 + i\)
   b. \(2i\)
   c. \(2 + 2i\)
   d. \(-\frac{9}{2} + \frac{15}{2}i\)
20. a. \(15 + 25i\)
   b. \(-2 + 21i\)
   c. \(289 + 0i\)
   d. \(20 + 40i\)
21. a. \(0 + i\)
   b. \(-\frac{26}{25} + \frac{7}{25}i\)
   c. \(-3 - 7i\)
22. \((2 + 4i)^2 - 4(2 + 4i) + 20 = 0\)
   \(4 + 8i + 16i^2 - 8 - 16i + 20 = 0\)
   \((4 - 16 - 8 + 20) + (8i + 8i - 16i) = 0\)
   \(0 + 0i = 0\)
   \(0 = 0\)
   \((2 - 4i)^2 - 4(2 - 4i) + 20 = 0\)
   \(4 - 8i - 16i^2 - 8 + 16i + 20 = 0\)
   \((4 - 16 - 8 + 20) + (-8i - 8i + 16i) = 0\)
   \(0 + 0i = 0\)
   \(0 = 0\)
23. \(2a\)
24. Sample answer: Use the Commutative Property to write the sum \(5 + 8i + 6 + 2i\) as \(5 + 6 + 8i + 2i\). Then use the Associative Property to group the real addends and the imaginary addends: \((5 + 6) + (8i + 2i)\). Add the real addends, and then use the Distributive Property to add the imaginary addends: \(11 + (8 + 2)i = 11 + 10i\).

ADAPT

Check students' answers to the Lesson Practice to ensure that they have mastered arithmetic operations involving complex numbers. In addition, make sure that students have a thorough understanding of complex conjugates. Provide additional practice with operations on complex numbers by having students solve quadratic equations with complex solutions and checking the solutions through substitution.
Lesson 8-3

Factoring with Complex Numbers

Learning Targets:
- Factor quadratic expressions using complex conjugates.
- Solve quadratic equations with complex roots by factoring.

**SUGGESTED LEARNING STRATEGIES:** Discussion Groups, Look for a Pattern, Quickwrite, Self Revision/Peer Revision, Paraphrasing

1. Look back at your answer to Item 13 in the previous lesson.
   a. Given your answer, what are the factors of the expression $a^2 + b^2$?
   Justify your answer.
   $a + bi$ and $a - bi$. Sample explanation: Multiply the factors: $a^2 + b^2$:
   $(a + bi)(a - bi) = a^2 - abi + abi - b^2 = a^2 - b^2(-1) = a^2 + b^2$
   b. What is the relationship between the factors of $a^2 + b^2$?
   They are complex conjugates.

You can use complex conjugates to factor quadratic expressions that can be written in the form $a^2 + b^2$. In other words, you can use complex conjugates to factor the sum of two squares.

2. Express regularity in repeated reasoning. Use complex conjugates to factor each expression.
   a. $16x^2 + 25$
   $(4x)^2 + 5^2 = (4x + 5)(4x - 5)$
   b. $36x^2 + 100y^2$
   $4(9x^2 + 25y^2) = 4((3x)^2 + (5y)^2) = 4(3x + 5y)(3x - 5y)$
   c. $2x^2 + 8y^2$
   $2(x^2 + 4y^2) = 2(x^2 + (2y)^2) = 2(x + 2yi)(x - 2yi)$
   d. $3x^2 + 20y^2$
   $(x\sqrt{3})^2 + (2y\sqrt{5})^2 = (x\sqrt{3} + 2yi\sqrt{5})(x\sqrt{3} - 2yi\sqrt{5})$
Lesson 8-3
Factoring with Complex Numbers

Check Your Understanding

3. Explain how to factor the expression $81x^2 + 64$.
4. Compare and contrast factoring an expression of the form $a^2 - b^2$ and an expression of the form $a^2 + b^2$.
5. Critique the reasoning of others. A student incorrectly claims that the factored form of the expression $4x^2 + 5$ is $(4x + 5)(4x - 5)$.
   a. Describe the error that the student made.
   b. How could the student have determined that his or her answer is incorrect?
   c. What is the correct factored form of the expression?

You can solve some quadratic equations with complex solutions by factoring.

Example A
Solve $9x^2 + 16 = 0$ by factoring.

Original equation
$9x^2 + 16 = 0$

Step 1: Factor the left side.
$(3x + 4)(3x - 4) = 0$

Step 2: Apply the Zero Product Property.
$3x + 4 = 0$ or $3x - 4 = 0$

Step 3: Solve each equation for $x$.

Solution: $x = -\frac{4}{3}$ or $x = \frac{4}{3}$

Try These A

a. Solve $x^2 + 81 = 0$ and check by substitution.

<table>
<thead>
<tr>
<th>$x^2 + 81 = 0$</th>
<th>Original equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x + 9)(x - 9) = 0$</td>
<td>Factor the left side.</td>
</tr>
<tr>
<td>$x + 9 = 0$ or $x - 9 = 0$</td>
<td>Apply the Zero Product Property.</td>
</tr>
<tr>
<td>$x = -9i$ or $x = 9i$</td>
<td>Solve each equation for $x$.</td>
</tr>
</tbody>
</table>

Solve each equation by factoring. Show your work.

b. $100x^2 + 49 = 0$
   $x = -\frac{7}{10}$, $x = \frac{7}{10}$

c. $25x^2 = -4$
   $x = -\frac{2}{5i}$, $x = \frac{2}{5i}$

d. $2x^2 + 36 = 0$
   $x = -3i\sqrt{2}$, $x = 3i\sqrt{2}$

e. $4x^2 = -45$
   $x = -\frac{3\sqrt{5}}{2}i$, $x = \frac{3\sqrt{5}}{2}i$

ACTIVITY 8
continued

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to factoring quadratic expressions.

Answers

3. Sample answer: The terms have no common factors, so start by writing the expression in the form $a^2 + b^2$: $(9x)^2 + 8^2$. Then factor the sum of squares by writing it in the form $(a + bi)(a - bi)$: $(9x)^2 + 8^2 = (9x + 8i)(9x - 8i)$.

4. Sample answer: The expression $a^2 - b^2$ is a difference of squares, and its factored form is $(a + b)(a - b)$. The expression $a^2 + b^2$ is a sum of squares, and its factored form is $(a + bi)(a - bi)$. The factored form of $a^2 - b^2$ is the product of 2 binomials. The first terms of each binomial are the same, and the second terms are opposites. The factored form of $a^2 + b^2$ is the product of 2 complex numbers. The real parts of each complex number are the same, and the imaginary parts are opposites.

5. a. The student factored an expression having the form $a^2x^2 + b^2$ as $(ax + bi)(ax - bi)$, instead of correctly factoring it as $(ax + bi)(ax - bi)$.

   b. The student could have used multiplication to find that $(4x + 5i)(4x - 5i)$ is equal to $16x^2 + 25$ instead of $4x^3 + 5i$.

   c. $(2x + i\sqrt{5})(2x - i\sqrt{5})$

Example A Debriefing

In the Try

These A, Item b is very straightforward, written as a sum of two squares, $a$ being $10x$ and $b$ being 7. Item c requires the student to add 4 to both sides to make it a sum of two squares equal to zero, where $a$ is 5x and $b$ is 2. Item d is a little more complex because students may or may not factor out the GCF of 2. The solution will be the same either way, but the steps will look very different, so be prepared to address this. If students do not factor out a common factor of 2, they should have $(x\sqrt{2} + 6i)(x\sqrt{2} + 6i) = 0$, for which they will have to rationalize a denominator in the process of isolating $x$. If they do factor out a common factor of 2, they should have $x^2 + 18 = 0$ to solve, where $a = x$ and $b = \sqrt{18}$, or $3\sqrt{2}$. Item e requires students to add 45 to both sides to make it a sum of two squares equal to zero, where $a = 2x$ and $b = 3\sqrt{5}$.
ACTIVITY 8 continued

Check Your Understanding
Debrief students’ answers to these items to ensure that they understand concepts related to the types of solutions of quadratic equations.

Answers
6. a. Real solutions; Sample explanation: The left side of the equation can be factored as a difference of squares: 
\((x + 12)(x - 12) = 0\). The solutions of the equation are \(x = -12\) and \(x = 12\), which are both real.
b. Imaginary solutions; Sample explanation: The left side of the equation can be factored as a sum of squares: 
\((x + 12i)(x - 12i) = 0\). The solutions of the equation are \(x = -12i\) and \(x = 12i\), which are both imaginary.

7. a. \(x = \frac{-b}{2a}i\) and \(x = \frac{b}{2a}i\). The left side of the equation can be factored as a sum of squares: 
\((ax + bi)(ax - bi) = 0\). Apply the Zero Product Property: 
\(ax + bi = 0\) or \(ax - bi = 0\). Then solve each equation for \(x\): 
\[x = \frac{-b}{2a}i\] and \[x = \frac{b}{2a}i\]. 
b. Answers may vary but should be one of the following: The solutions are opposites. The solutions are additive inverses. The solutions are complex conjugates.

8. Sample answer: Substitute 0 for \(f(x)\): 
\[0 = x^2 + 225\]. Factor the right side as a sum of squares: 
\[0 = (x + 15i)(x - 15i)\]. Then solve for \(x\): 
\[x = -15i\] or \(x = 15i\). The solutions of the function are \(-15i\) and \(15i\).

LESSON 8-3 PRACTICE

Check Your Understanding
6. Tell whether each equation has real solutions or imaginary solutions and explain your answer.
   a. \(x^2 - 144 = 0\) b. \(x^2 + 144 = 0\)
7. a. What are the solutions of a quadratic equation that can be written in the form \(ax^2 + bx = 0\), where \(a\) and \(b\) are real numbers and \(a \neq 0\)? Show how you determined the solutions.
b. What is the relationship between the solutions of a quadratic equation that can be written in the form \(ax^2 + bx = 0\)?
8. Explain how you could find the solutions of the quadratic function \(f(x) = x^2 + 225\) when \(f(x) = 0\).

Use complex conjugates to factor each expression.
9. \(3x^2 + 12\) 10. \(5x^2 + 80i\)
11. \(9x^2 + 11\) 12. \(2x^2 + 63y^2\)
Solve each equation by factoring.
13. \(2x^2 + 50 = 0\) 14. \(3x^2 = -54\)
15. \(4x^2 + 75 = 0\) 16. \(32x^2 = -98\)
17. Reason quantitatively. Solve the equations \(9x^2 - 64 = 0\) and \(9x^2 + 64 = 0\) by factoring. Then describe the relationship between the solutions of \(9x^2 - 64 = 0\) and the solutions of \(9x^2 + 64 = 0\).

LESSON 8-3 PRACTICE

9. \(3(x + 2i)(x - 2i)\)
10. \(5(x + 4y)(x - 4y)\)
11. \((3x + i\sqrt{11})(3x - i\sqrt{11})\)
12. \((x\sqrt{2} + 3yi\sqrt{7})(x\sqrt{2} - 3yi\sqrt{7})\)
13. \(x = -5i, x = 5i\)
14. \(x = -3i\sqrt{2}, x = 3i\sqrt{2}\)
15. \(x = \frac{5\sqrt{3}i}{2}, x = \frac{-5\sqrt{3}i}{2}\)
16. \(x = -\frac{7}{4}i, x = \frac{7}{4}i\)
17. The solutions of \(9x^2 - 64 = 0\) are \(x = -\frac{8}{3}\) and \(x = \frac{8}{3}\). The solutions of 
\[9x^2 + 64 = 0\] are \(x = -\frac{8}{3}i\) and \(x = \frac{8}{3}i\).

The solutions of \(9x^2 - 64 = 0\) are real, and the solutions of \(9x^2 + 64 = 0\) are imaginary. The real parts of the solution of \(9x^2 - 64 = 0\) are the same as the imaginary parts of the solutions of \(9x^2 + 64 = 0\).
ACTIVITY 8 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 8-1

1. Write each expression in terms of \(i\).
   a. \(\sqrt{-64}\)
   b. \(\sqrt{-31}\)
   c. \(-7 + \sqrt{-12}\)
   d. \(5 - \sqrt{-50}\)

2. Which expression is equivalent to \(5i\)?
   A. \(\sqrt{-5}\)
   B. \(-\sqrt{5}\)
   C. \(\sqrt{-25}\)
   D. \(-\sqrt{25}\)

3. Use the Quadratic Formula to solve each equation.
   a. \(x^2 + 5x + 9 = 0\)
   b. \(2x^2 - 4x + 5 = 0\)

4. The sum of two numbers is 12, and their product is 100.
   a. Let \(x\) represent one of the numbers. Write an expression for the other number in terms of \(x\).
      Use the expressions to write an equation that models the situation given above.
   b. Use the Quadratic Formula to solve the equation. Write the solutions in terms of \(i\).

5. Explain why each of the following is a complex number, and identify its real part and its imaginary part.
   a. \(5 + 3i\)
   b. \(\sqrt{2} - i\)
   c. \(-14i\)
   d. \(\frac{3}{4}\)

6. Draw the complex plane on grid paper. Then graph each complex number on the plane.
   a. \(-4i\)
   b. \(6 + 2i\)
   c. \(-3 - 4i\)
   d. \(3 - 5i\)
   e. \(-2 + 5i\)

7. What complex number does the ordered pair \((5, -3)\) represent on the complex plane? Explain.

8. Name the complex number represented by each labeled point on the complex plane below.

   ![Complex Plane Diagram]

Lesson 8-2

9. Find each sum or difference.
   a. \((5 - 6i) + (-3 + 9i)\)
   b. \((2 + 5i) + (-5 + 3i)\)
   c. \((9 - 2i) - (1 + 6i)\)
   d. \((-5 + 4i) - \left(\frac{7}{3} + \frac{1}{6}\right)\)

10. Find each product, and write it in the form \(a + bi\).
    a. \((1 + 4i)(5 - 2i)\)
    b. \((-2 + 3i)(3 - 2i)\)
    c. \((7 + 24i)(7 - 24i)\)
    d. \((8 - 3i)(4 - 2i)\)

11. Find each quotient, and write it in the form \(a + bi\).
    a. \(\frac{3 + 2i}{5 - 2i}\)
    b. \(\frac{-1 + i}{5 - 2i}\)
    c. \(\frac{10 - 2i}{5i}\)
    d. \(\frac{3 + i}{3 - i}\)

12. Explain how to use the Commutative, Associative, and Distributive Properties to perform each operation.
    a. Subtract \((3 + 4i)\) from \((8 + 5i)\).
    b. Multiply \((-2 + 3i)\) and \((4 - 6i)\).

   ![Complex Plane Diagram]

7. \(5 - 3i\): The first number in the ordered pair is the real part of the complex number, and the second number in the ordered pair is the imaginary part of the complex number.

8. a. \(-3 + 2i\)
   b. \(2 + 5i\)
   c. \(3\)
   d. \(4 - 3i\)
   e. \(-5 - 2i\)

9. a. \(2 + 3i\)
   b. \(3 - 8i\)
   c. \(-8 - 3i\)
   d. \(-\frac{22}{3} - \frac{23}{6}\)

10. a. \(-13 + 18i\)
    b. \(0 + 13i\)
    c. \(625 + 0i\)
    d. \(26 - 28i\)
13. Accept any complex number with a real part of -4. Sample answer: -4 + 2i; (4 - 8i) + (-4 + 2i) = 0 - 6i = -6i, and -6i is an imaginary number.

14. If students need more practice on the squares:

a. $x^2 - 6x + 34 = 0$

b. $(9 - 15i - 15i + 30i) = 0$

c. $0 + 0i = 0$

d. $0 = 0$

15. Simplify each expression.

a. $-i^2$

b. $-6i^4$

c. $(2i)^3$

d. $\left(\frac{3}{2}\right)^2$

16. What is the difference of any complex number $a + bi$ and its complex conjugate?

17. Use substitution to show that the solutions of the equation $x^2 - 6x + 34 = 0$ are $x = 3 + 5i$ and $x = 3 - 5i$.

18. a. Graph the complex number $4 + 2i$ on a complex plane.

b. Multiply $4 + 2i$ by $i$, and graph the result.

c. Multiply the result from part b by $i$, and graph the result.

d. Multiply the result from part c by $i$, and graph the result.

e. Describe any patterns you see in the complex numbers you graphed.

f. What happens when you multiply a complex number $a + bi$ by $i$?

19. a. $x^2 + 121$

b. $2x^2 + 128y^2$

c. $4x^2 + 60y^2$

d. $9x^2 + 140y^2$

20. Explain how to solve the equation $2x^2 + 100 = 0$ by factoring.

21. Solve each equation by factoring.

a. $x^2 + 64 = 0$

b. $x^2 = -120$

c. $4x^2 + 169 = 0$

d. $25x^2 = -48$

22. Which equation has solutions of $x = \frac{-2}{3}$ and $x = \frac{2}{3}$?

23. What are the solutions of each quadratic function?

a. $f(x) = x^2 + 1$

b. $f(x) = 25x^2 + 36$

24. Without solving the equation, explain how you know that $x^2 + 48 = 0$ has imaginary solutions.

**Mathematical Practices**

Look for and Express Regularity in Repeated Reasoning

25. Find the square of each complex number.

a. $(4 + 5i)$

b. $(2 + 3i)$

c. $(4 - 2i)$

d. Use parts a–c and your knowledge of operations of real numbers to write a general formula for the square of a complex number $(a + bi)$.

20. Factor 2 from both terms of the left side:

$2(x^2 + 50) = 0$. Write $x^2 + 50$ as a sum of squares: $x^2 + (\sqrt{50})^2 = 0$. Factor the left side: $2(\frac{x + 5\sqrt{2}}{2})(\frac{x - 5\sqrt{2}}{2}) = 0$. Solve for $x$:

$x = -5\sqrt{2} \text{ or } x = 5\sqrt{2}$.

21. a. $x = -8i$; $x = 8i$

b. $x = -2i\sqrt{50}$; $x = 2i\sqrt{50}$

c. $x = -\frac{13}{2}$; $x = \frac{13}{2}$

d. $x = -\frac{4\sqrt{3}}{5}$; $x = \frac{4\sqrt{3}}{5}$

22. D

23. a. $-i$ and $i$

b. $\frac{6}{5}i$ and $\frac{6}{5}i$

24. Sample answer: If you subtract 48 from both sides of the equation, you get $x^2 = -48$. To solve for $x$, you must find the positive and negative square roots of a negative number which are imaginary, so the solutions are imaginary.

25. a. $(4 + 5i)^2 = -9 + 40i$

b. $(2 + 3i)^2 = -5 + 12i$

c. $(4 - 2i)^2 = 12 - 16i$

d. $(a + bi)^2 = (a^2 - b^2) + 2abi$
Solving $ax^2 + bx + c = 0$

**Deriving the Quadratic Formula**

**Lesson 9-1 Completing the Square and Taking Square Roots**

### Learning Targets:
- Solve quadratic equations by taking square roots.
- Solve quadratic equations $ax^2 + bx + c = 0$ by completing the square.

**SUGGESTED LEARNING STRATEGIES:** Marking the Text, Group Presentation, Quickwrite, Create Representations

To solve equations of the form $ax^2 + c = 0$, isolate $x^2$ and take the square root of both sides of the equation.

#### Example A

Solve $5x^2 - 23 = 0$ for $x$.

**Step 1:** Add 23 to both sides.  \[ 5x^2 = 23 \]

**Step 2:** Divide both sides by 5.  \[ 5 \]

**Step 3:** Simplify to isolate $x^2$.  \[ x^2 = \frac{23}{5} \]

**Step 4:** Take the square root of both sides.  \[ x = \pm \sqrt{\frac{23}{5}} \]

**Step 5:** Rationalize the denominator.  \[ x = \pm \frac{\sqrt{115}}{\sqrt{5}} \]

**Step 6:** Simplify.  \[ x = \pm \frac{\sqrt{115}}{5} \]

**Solution:** \[ x = \pm \frac{\sqrt{115}}{5} \]

#### Try These A

**Make use of structure.** Solve for $x$. Show your work.

<table>
<thead>
<tr>
<th>Example</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. [9x^2 - 49 = 0]</td>
<td>[x = \frac{7}{3}, x = -\frac{7}{3}]</td>
</tr>
<tr>
<td>b. [25x^2 - 7 = 0]</td>
<td>[x = \frac{\sqrt{7}}{5}, x = -\frac{\sqrt{7}}{5}]</td>
</tr>
<tr>
<td>c. [5x^2 - 16 = 0]</td>
<td>[x = \frac{4\sqrt{5}}{5}, x = -\frac{4\sqrt{5}}{5}]</td>
</tr>
<tr>
<td>d. [4x^2 + 15 = 0]</td>
<td>[x = \frac{i\sqrt{15}}{2}, x = -\frac{i\sqrt{15}}{2}]</td>
</tr>
</tbody>
</table>

**MATH TIP**

When taking the square root of both sides of an equation, include both positive and negative roots. For example, \[x^2 = 4\]

\[x = \pm \sqrt{4}\]

\[x = \pm 2\]

**MATH TIP**

To rewrite an expression so that there are no radicals in the denominator, you must **rationalize the denominator** by multiplying both the numerator and denominator by the radical.

**Example:**

\[\frac{7}{\sqrt{3}} = \frac{7\sqrt{3}}{3}\]

**Common Core State Standards for Activity 9**

**HSN-CN.C.7** Solve quadratic equations with real coefficients that have complex solutions.
Lesson 9-1
Completing the Square and Taking Square Roots

1. Compare and contrast the solutions to the equations in Try These A.
   Sample answers:
   Parts a, b, and c are real-number solutions; part d is complex.
   Part a is rational; parts b and c are irrational.
   Parts b and d did not need to be rationalized; part c was rationalized.
   To solve the equation $2(x - 3)^2 - 5 = 0$, you can use a similar process.

   **Example B**
   Solve $2(x - 3)^2 - 5 = 0$ for $x$.
   
   Step 1: Add 5 to both sides. $2(x - 3)^2 = 5$
   
   Step 2: Divide both sides by 2. $(x - 3)^2 = \frac{5}{2}$
   
   Step 3: Take the square root of both sides. $x - 3 = \pm \sqrt{\frac{5}{2}}$
   
   Step 4: Rationalize the denominator and solve for $x$. $x = 3 \pm \frac{\sqrt{10}}{2}$
   
   Solution: $x = 3 \pm \frac{\sqrt{10}}{2}$

   **Try These B**
   Solve for $x$. Show your work.
   
   a. $4(x + 5)^2 - 49 = 0$
      
   b. $3(x - 2)^2 - 16 = 0$
      
   c. $5(x + 1)^2 - 8 = 0$
      
   d. $4(x + 7)^2 + 25 = 0$

   **2. Reason quantitatively.** Describe the differences among the solutions to the equations in Try These B.
   
   Sample answers:
   Parts a, b, and c are real-number solutions; part d is complex.
   Part a is a rational solution; parts b and c are irrational.
   Parts b and c were rationalized.
Lesson 9-1
Completing the Square and Taking Square Roots

Check Your Understanding

3. Use Example A to help you write a general formula for the solutions of the equation \( ax^2 - c = 0 \), where \( a \) and \( c \) are both positive.
4. Is the equation solved in Example B a quadratic equation? Explain.
5. Solve the equation \(-2(x + 4)^2 = 3\), and explain each of your steps.
6. a. Solve the equation \( 3(x - 5)^2 = 0 \).
   b. Make use of structure. Explain why the equation has only one solution and not two solutions.

The standard form of a quadratic equation is \( ax^2 + bx + c = 0 \). You can solve equations written in standard form by completing the square.

Example C
Solve \( 2x^2 + 12x + 5 = 0 \) by completing the square.

Step 1: Divide both sides by the leading coefficient and simplify.
\[
\frac{2x^2 + 12x + 5}{2} = \frac{0}{2}
\]
\[
x^2 + 6x + \frac{5}{2} = 0
\]

Step 2: Isolate the variable terms on the left side.
\[
x^2 + 6x = -\frac{5}{2}
\]

Step 3: Divide the coefficient of the linear term by 2 [6 ÷ 2 = 3], square the result \( \left(3^2 = 9\right) \), and add it to both sides. This completes the square.
\[
x^2 + 6x + 9 = -\frac{5}{2} + 9
\]
\[
x^2 + 6x + 9 = \frac{7}{2}
\]

Step 4: Factor the perfect square trinomial on the left side into two binomials.
\[
(x + 3)^2 = \frac{13}{2}
\]

Step 5: Take the square root of both sides of the equation.
\[
x + 3 = \pm \sqrt{\frac{13}{2}}
\]
\[
x + 3 = \pm \frac{\sqrt{26}}{2}
\]

Solution: \( x = -3 \pm \frac{\sqrt{26}}{2} \)

Math Terms

Completing the square is the process of adding a constant to a quadratic expression to transform it into a perfect square trinomial.

MATH TIP

You can factor a perfect square trinomial \( x^2 + 2xy + y^2 \) as \( (x + y)^2 \).

Check Your Understanding

Debrief Students’ Answers to These Items to Ensure That They Understand Concepts Related to Quadratic Functions and Solving Quadratic Equations by Taking Square Roots.

Developing Math Language

The process of completing the square is a series of steps that can be used to solve any quadratic equation. The process transforms a quadratic expression into a perfect square trinomial that factors into the square of a binomial. Here is a summary of the steps shown in Example C for the standard form of a quadratic expression \( ax^2 + bx + c = 0 \):

1. Divide both sides of the equation \( ax^2 + bx + c = 0 \) by \( a \) (when \( a \neq 0 \)) and then subtract \( \frac{c}{a} \) from both sides.
2. Now take the value of the coefficient of \( x \left( \frac{b}{a} \right) \) or \( b \) if \( a = 1 \), divide it by 2, and square it, to get \( \left( \frac{b}{2a} \right)^2 \). Add this value to both sides of the equation. This is when you are actually completing the square. The result is a perfect square trinomial on the left side of the equation:
\[
x^2 + \frac{b}{a}x + \left( \frac{b}{2a} \right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}
\]
3. Factor the perfect square trinomial you have just created (refer to the Math Tip alongside Example C). The factored form of the left side of the equation is \( x + \left( \frac{b}{2a} \right)^2 \).
4. Take the square root of both sides of the equation, and simplify and rationalize as necessary. These final steps will be covered in more detail in the next lesson when students derive the quadratic formula.

Example C Marking the Text, Activating Prior Knowledge, Create Representations, Debriefing

The method utilized in the example is used to derive the quadratic formula, allowing students an opportunity for success in Item 1 of the next lesson. Students must exercise care to ensure that equivalent quantities exist on both sides of the equation.

Again, rationalizing the denominator is essential to obtain solutions that are the same in format as those obtained by using the Quadratic Formula. It may be worthwhile to emphasize the meaning of the words completing the square. Essentially, the goal of the process is to obtain an equation that can be solved by taking the square root of both sides of the equation. That can only occur if there is a perfect square on one side—hence the need to complete the perfect square trinomial by the addition of the appropriate constant term.
ACTIVITY 9 Continued

Check Your Understanding
Debrief students’ answers to these items to ensure that they understand concepts related to quadratic functions and solving quadratic equations by completing the square.

Answers
7. Divide the coefficient of the x-term by 2: \(8 \div 2 = 4\). Next, square the result: \(4^2 = 16\). Then, add the final result to the quadratic expression: \(x^2 + 8x + 16\).

8. Completing the square lets you write one side of the quadratic equation as a perfect square trinomial. After you factor the perfect square trinomial, you can solve the equation by taking the square root of both sides.

9. Sample answer: I would solve the equation by factoring. I can tell by using mental math that the factors of –12 that have a sum of 1 are 4 and –3. The equation can be factored as \((x + 4)(x – 3) = 0\), which means that its solutions are \(x = –4\) and \(x = 3\). If I solved the equation by completing the square, I would need to perform many more steps: isolating the variable terms, completing the square, factoring the perfect square trinomial, taking the square root of both sides, and then solving for \(x\).

LESSON 9-1 PRACTICE

10. Use the method for completing the square to make a perfect square trinomial. Then factor the perfect square trinomial.
   a. \(x^2 + 10x\)  
   b. \(x^2 + 2x\)

11. Solve each quadratic equation by taking the square root of both sides of the equation. Identify the solutions as rational, irrational, or complex conjugates.
   a. \(9x^2 – 64 = 0\)  
   b. \(5x^2 – 12 = 0\)
   c. \(16(x – 2)^2 – 25 = 0\)  
   d. \(2(x – 3)^2 – 15 = 0\)
   e. \(4x^2 + 49 = 0\)  
   f. \(3(x – 1)^2 + 10 = 0\)

12. Solve by completing the square.
   a. \(x^2 – 4x – 12 = 0\)  
   b. \(2x^2 – 5x – 3 = 0\)
   c. \(x^2 + 6x – 2 = 0\)  
   d. \(3x^2 + 9x + 2 = 0\)
   e. \(x^2 – x + 5 = 0\)  
   f. \(5x^2 + 2x + 3 = 0\)

13. The diagonal of a rectangular television screen measures 42 in. The ratio of the length to the width of the screen is \(\frac{16}{9}\).
   a. Model with mathematics. Write an equation that can be used to determine the length \(l\) in inches of the television screen.
   b. Solve the equation, and interpret the solutions.
   c. What are the length and width of the television screen, to the nearest half-inch?

Try These C
Solve for \(x\) by completing the square.

\(a. 4x^2 + 16x – 5 = 0\)  
\(b. 5x^2 – 30x – 3 = 0\)
\(x = -2 \pm \frac{\sqrt{21}}{2}\)  
\(x = 3 \pm \frac{\sqrt{15}}{5}\)
\(c. 2x^2 – 6x – 1 = 0\)  
\(d. 2x^2 – 4x + 7 = 0\)
\(x = 2 \pm \frac{\sqrt{11}}{2}\)  
\(x = 1 \pm \frac{\sqrt{10}}{2}\)

Check Your Understanding
7. Explain how to complete the square for the quadratic expression \(x^2 + 8x\).
8. How does completing the square help you solve a quadratic equation?
9. Construct viable arguments. Which method would you use to solve the quadratic equation \(x^2 + x – 12 = 0\): factoring or completing the square? Justify your choice.

LESSON 9-1 PRACTICE

10. a. \(x^2 + 10x\)  
    b. \(x^2 + 7x\)

11. Solve each quadratic equation by taking the square root of both sides of the equation. Identify the solutions as rational, irrational, or complex conjugates.
    a. \(9x^2 – 64 = 0\)  
    b. \(5x^2 – 12 = 0\)
    c. \(16(x – 2)^2 – 25 = 0\)  
    d. \(2(x – 3)^2 – 15 = 0\)
    e. \(4x^2 + 49 = 0\)  
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    b. \(2x^2 – 5x – 3 = 0\)
    c. \(x^2 + 6x – 2 = 0\)  
    d. \(3x^2 + 9x + 2 = 0\)
    e. \(x^2 – x + 5 = 0\)  
    f. \(5x^2 + 2x + 3 = 0\)

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Try These C
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\(a. 4x^2 + 16x – 5 = 0\)  
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\(x = 3 \pm \frac{\sqrt{15}}{5}\)
\(c. 2x^2 – 6x – 1 = 0\)  
\(d. 2x^2 – 4x + 7 = 0\)
\(x = 2 \pm \frac{\sqrt{11}}{2}\)  
\(x = 1 \pm \frac{\sqrt{10}}{2}\)
Lesson 9-2
The Quadratic Formula

Learning Targets:
• Derive the Quadratic Formula.
• Solve quadratic equations using the Quadratic Formula.

SUGGESTED LEARNING STRATEGIES: Create Representations, Discussion Groups, Self Revision/Peer Revision, Think-Pair-Share, Quickwrite

Previously you learned that solutions to the general quadratic equation $ax^2 + bx + c = 0$ can be found using the Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a \neq 0$$

You can derive the quadratic formula by completing the square on the general quadratic equation.

1. **Reason abstractly and quantitatively.** Derive the quadratic formula by completing the square for the equation $ax^2 + bx + c = 0$.

   (Use Example C from Lesson 9-1 as a model.)

   If $ax^2 + bx + c = 0$

   $$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

   $$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

   $$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

   $$x + \frac{b}{2a} = \pm \sqrt{\frac{b^2 - 4ac}{4a^2}} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

   $$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} = -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

**ACADEMIC VOCABULARY**

When you derive a formula, you use logical reasoning to show that the formula is correct. In this case, you will derive the Quadratic Formula by solving the standard form of a quadratic equation, $ax^2 + bx + c = 0$, for $x$.

**Plan**

Pacing: 1 class period

Chunking the Lesson

#1–2
Check Your Understanding
Lesson Practice

**Teach**

- **Bell-Ringer Activity**

Have students solve the following items using the method given.

1. By factoring: $x^2 - 2x - 15 = 0$  \[x = 5, -3\]
2. By completing the square: $x^2 - 2x - 5 = 0$  \[x = 1 \pm \sqrt{6}\]
3. By using the quadratic formula: $3x^2 + 7x - 20 = 0$  \[x = \frac{5}{3}, -4\]

This will provide students with a brief overview of the various methods that can be used to solve a quadratic equation.

1–2 Create Representations, Group Presentations, Debriefing

The derivation of the quadratic formula is a nontrivial task for most students. A student who masters this easily is likely one who has mastered the abstraction of algebraic methods. Many students may struggle with the derivation, so monitor student progress closely. A whole-class discussion and debriefing following the allotted time period is a critical strategy.

ELL Support

In this activity, the word derive is used. By applying the method of completing the square to the general equation $ax^2 + bx + c = 0$, we are able to trace the steps that it takes to arrive at the quadratic formula. Students should follow these steps so that they have a better understanding of how the quadratic formula originated and why it can be used to solve any quadratic equation.
Lesson 9-2
The Quadratic Formula

2. Solve \(2x^2 - 5x + 3 = 0\) by completing the square. Then verify that the solution is correct by solving the same equation using the Quadratic Formula.

   **Completing the square:**
   
   \[2x^2 - 5x + 3 = 0\]
   
   \[x^2 - \frac{5}{2}x + \frac{3}{2} = 0\]
   
   \[x^2 - \frac{5}{2}x + \frac{25}{16} = -3 + \frac{25}{16}\]
   
   \[\left(x - \frac{5}{4}\right)^2 = \frac{1}{16}\]
   
   \[x - \frac{5}{4} = \pm \frac{1}{4}\]
   
   therefore, \(x = \frac{5}{4} + \frac{1}{4} = \frac{3}{2}\) or \(x = \frac{5}{4} - \frac{1}{4} = 1\)

   **Using the Quadratic Formula:**
   
   \[a = 2, \ b = -5, \ c = 3, \ \text{therefore} \ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\]
   
   \[x = \frac{5 \pm \sqrt{25 - 24}}{4} = \frac{5 \pm 1}{4}\]
   
   therefore, \(x = \frac{5}{4} + \frac{1}{4} = 1\) or \(x = \frac{5}{4} - \frac{1}{4} = \frac{3}{2}\)

3. In Item 1, why do you need to add \(\left(\frac{b}{2a}\right)^2\) to both sides?

4. Derive a formula for solving a quadratic equation of the form \(ax^2 + bx = 0\), where \(a \neq 0\).

5. Construct viable arguments. Which method did you prefer for solving the quadratic equation in Item 2: completing the square or using the Quadratic Formula? Justify your choice.

6. Consider the equation \(x^2 - 6x + 7 = 0\).
   
   a. Solve the equation by using the Quadratic Formula.
   
   b. Could you have solved the equation by factoring? Explain.
Lesson 9-2
The Quadratic Formula

**LESSON 9-2 PRACTICE**

7. Solve each equation by using the Quadratic Formula.
   a. $2x^2 + 4x - 5 = 0$
   b. $3x^2 + 7x + 10 = 0$
   c. $x^2 - 9x - 1 = 0$
   d. $-4x^2 + 5x + 8 = 0$
   e. $2x^2 - 3 = 7x$
   f. $4x^2 + 3x = -6$

8. Solve each quadratic equation by using any of the methods you have learned. For each equation, tell which method you used and why you chose that method.
   a. $x^2 + 6x + 9 = 0$
   b. $8x^2 + 5x - 6 = 0$
   c. $(x + 4)^2 - 36 = 0$
   d. $x^2 + 2x = 7$

9. **a. Reason abstractly.** Under what circumstances will the radicand in the Quadratic Formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, be negative?
   **b.** If the radicand is negative, what does this tell you about the solutions of the quadratic equation? Explain.

10. A player shoots a basketball from a height of 7 ft with an initial vertical velocity of 18 ft/s. The equation $-16t^2 + 18t + 7 = 10$ can be used to determine the time $t$ in seconds at which the ball will have a height of 10 ft—the same height as the basket.
   **a.** Solve the equation by using the Quadratic Formula.
   **b.** **Attend to precision.** To the nearest tenth of a second, when will the ball have a height of 10 ft?
   **c.** Explain how you can check that your answers to part b are reasonable.

---

**ACTIVITY 9**

**CONNECT TO PHYSICS**

The function $h(t) = -16t^2 + v_0t + h_0$ can be used to model the height $h$ in feet of a thrown object $t$ seconds after it is thrown, where $v_0$ is the initial vertical velocity of the object in ft/s and $h_0$ is the initial height of the object in feet.

**MATH TIP**

A radicand is an expression under a radical symbol. For $\sqrt{b^2 - 4ac}$, the radicand is $b^2 - 4ac$. $c.$ Sample answer: Substitute the times from part $b$ into the original equation to check that they make the left side of the equation approximately equal to 10.

- $-16(0.2)^2 + 18(0.2) + 7 \approx 10$
- $-0.64 + 3.6 + 7 \approx 10$
- $9.96 \approx 10 \checkmark$
- $-16(0.9)^2 + 18(0.9) + 7 \approx 10$
- $-12.96 + 16.2 + 7 \approx 10$
- $10.24 \approx 10 \checkmark$

---

**ADAPT**

Check students’ answers to the Lesson Practice to ensure that they understand how to solve a quadratic equation using the Quadratic Formula. Additionally, check to see that students are choosing and correctly using all possible methods for solving quadratic equations. Students may wish to create a graphic organizer of solution methods for quadratic equations to aid in their mastery of the methods.
Lesson 9-3

Solutions of Quadratic Equations

Learning Targets:
• Solve quadratic equations using the Quadratic Formula.
• Use the discriminant to determine the nature of the solutions of a quadratic equation.

SUGGESTED LEARNING STRATEGIES: Look for a Pattern, Group Presentation, Self Revision/Peer Revision, Think-Pair-Share, Quickwrite

1. Solve each equation by using the Quadratic Formula. For each equation, write the number of solutions. Tell whether the solutions are real or complex, and, if real, whether the solutions are rational or irrational.

a. \(4x^2 + 5x - 6 = 0\)
   solutions: \(x = -2, x = \frac{3}{4}\)
   number of solutions: \(2\)
   real or complex: real
   rational or irrational: irrational

b. \(4x^2 + 5x - 2 = 0\)
   solutions: \(x = \frac{-8 \pm \sqrt{57}}{8}\)
   number of solutions: \(2\)
   real or complex: real
   rational or irrational: irrational

c. \(4x^2 + 4x + 1 = 0\)
   solutions: \(x = -\frac{1}{2}\)
   number of solutions: \(1\)
   real or complex: real
   rational or irrational: irrational

d. \(4x^2 + 4x + 5 = 0\)
   solutions: \(x = -\frac{1}{2} \pm i\)
   number of solutions: \(2\)
   real or complex: complex
   rational or irrational: not applicable (since this applies to real numbers only)

The complex numbers include the real numbers, so real solutions are also complex solutions. However, when asked to classify solutions as real or complex, you can assume that “complex” does not include the reals.
Lesson 9-3
Solutions of Quadratic Equations

2. Express regularity in repeated reasoning. What patterns can you identify from your responses to Item 1?
   Patterns that students may recognize are that if \( b^2 - 4ac \) is positive, the solutions are real, and if \( b^2 - 4ac \) is negative, the solutions are complex.

Check Your Understanding

3. a. In Item 1, was the expression under the square root symbol of the Quadratic Formula positive, negative, or zero when there were two real solutions?
   b. What about when there was one real solution?
   c. What about when there were two complex solutions?
4. In Item 1, how did you determine whether the real solutions of a quadratic equation were rational or irrational?
5. Reason quantitatively. The quadratic function related to the equation in Item 1a is \( f(x) = 4x^2 + 5x - 6 \). Without graphing the function, determine how many \( x \)-intercepts it has and what their values are. Explain how you determined your answer.
6. Make a conjecture about the relationship between the solutions of a quadratic equation that has complex roots.

DIFFERENTIATING INSTRUCTION

In a problem such as Item 1d, a common error made by students occurs during the last step or steps of the solution. Some students have the misconception that with an expression such as \( \frac{-4 \pm 8i}{8} \), they can divide out the 8’s and simplify the fraction to \(-4 \pm i\). Students must realize that if you cancel one term of the numerator by a common factor, they must be able to divide all the terms of the numerator by that factor. In this case, the greatest factor they all have in common is 4.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to the nature of solutions of quadratic equations.

Answers
3. a. positive
   b. zero
   c. negative
4. Sample answer: If the solutions can be written as integers or as fractions with integer numerators and integer denominators, then solutions are rational. If the solutions involve square roots of non-perfect squares, then the solutions are irrational.
5. The function has two \( x \)-intercepts: \(-2\) and \(\frac{3}{4}\). Sample explanation: The \( x \)-intercepts are the real values of \( x \) for which \( f(x) = 0 \), so the \( x \)-intercepts are the real solutions of the equation \( 0 = 4x^2 + 5x - 6 \). Item 1a shows that this equation has two real solutions of \( x = -2 \) and \( x = \frac{3}{4} \), so the \( x \)-intercepts of the related function are \(-2\) and \(\frac{3}{4}\).
6. Sample answers: The solutions are complex conjugates. The solutions have the same real parts and opposite imaginary parts.
Developing Math Language

The expression under the radical sign of the quadratic formula, \( b^2 - 4ac \), is called the **discriminant**. The value of this expression enables one to determine, or **discriminate**, among the possible types of solutions of its corresponding quadratic equation.

Another word often used to refer to the solution(s) of an equation is **root(s)**. In the case of a quadratic equation of general form \( ax^2 + bx + c = 0 \), the root(s) refers to the zeros of the equation, or the location(s) where the graph of its function crosses the x-axis. The terms **root**, **zero**, and **solution** are all essentially synonymous.

### 7–8 Identify a Subtask, Debriefing

Students use the definition of the discriminant to evaluate and interpret earlier results. As students review the table, be sure they understand that \( a \), \( b \), and \( c \) must be rational numbers for the nature of the solutions described in the table to be true. For example, \( x^2 + 2x + \pi = 0 \) has an irrational solution of \( x \).

Evaluating the discriminant of the equations in Item 7 gives students the opportunity to formalize the conjectures that they may have put forth in their discussions.

**Note:** Be sure that students recognize the synonymous use of the words **solution** and **root**.

<table>
<thead>
<tr>
<th>Discriminant</th>
<th>Nature of Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b^2 - 4ac &gt; 0 ) and ( b^2 - 4ac ) is a perfect square</td>
<td>Two real, rational solutions</td>
</tr>
<tr>
<td>( b^2 - 4ac &gt; 0 ) and ( b^2 - 4ac ) is not a perfect square</td>
<td>Two real, irrational solutions</td>
</tr>
<tr>
<td>( b^2 - 4ac = 0 )</td>
<td>One real, rational solution (a double root)</td>
</tr>
<tr>
<td>( b^2 - 4ac &lt; 0 )</td>
<td>Two complex conjugate solutions</td>
</tr>
</tbody>
</table>

7. Compute the value of the discriminant for each equation in Item 1 to determine the number and nature of the solutions.

   a. \( 4x^2 + 5x - 6 = 0 \)

      121; Since \( b^2 - 4ac \) is positive and a perfect square, there are two real, rational roots.

   b. \( 4x^2 + 5x - 2 = 0 \)

      57; Since \( b^2 - 4ac \) is positive and not a perfect square, there are two real, irrational roots.

   c. \( 4x^2 + 4x + 1 = 0 \)

      0; Since \( b^2 - 4ac \) is zero, there is one real, rational root.

   d. \( 4x^2 + 4x + 5 = 0 \)

      -64; Since \( b^2 - 4ac \) is negative, there are two complex conjugate roots.
Lesson 9-3
Solutions of Quadratic Equations

8. For each equation below, compute the value of the discriminant and describe the solutions without solving.
   a. \(2x^2 + 5x + 12 = 0\)
      \[\text{discriminant} = -71; \text{two complex conjugate roots}\]
   b. \(3x^2 - 11x + 4 = 0\)
      \[\text{discriminant} = 73; \text{two real, irrational roots}\]
   c. \(5x^2 + 3x - 2 = 0\)
      \[\text{discriminant} = 49; \text{two real, rational roots}\]
   d. \(4x^2 - 12x + 9 = 0\)
      \[\text{discriminant} = 0; \text{one real, rational root}\]

Check Your Understanding

9. Critique the reasoning of others. A student solves a quadratic equation and gets solutions of \(x = -\frac{7}{3}\) and \(x = 8\). To check the reasonableness of his answer, the student calculates the discriminant of the equation and finds it to be \(-188\). Explain how the value of the discriminant shows that the student made a mistake when solving the equation.

10. One of the solutions of a quadratic equation is \(x = 6 + 4i\). What is the other solution of the quadratic equation? Explain your answer.

11. The discriminant of a quadratic equation is 225. Are the roots of the equation rational or irrational? Explain.

12. Consider the quadratic equation \(2x^2 + 5x + c = 0\).
   a. For what value(s) of \(c\) does the equation have two real solutions?
   b. For what value(s) of \(c\) does the equation have one real solution?
   c. For what value(s) of \(c\) does the equation have two complex conjugate solutions?
ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 9-3 PRACTICE

13. a. discriminant = 49, two rational roots, solutions are $x = -6$ and $x = 1$
   b. discriminant = 169, two rational roots, solutions are $x = -\frac{9}{2}$ and $x = 5$
   c. discriminant = 0, one rational (double) root, solution is $x = 4$
   d. discriminant = -24, two complex roots, solutions are $x = 2, 3 + 2i$
   e. discriminant = -79, two complex roots, solutions are $x = 9, 2 - 2i$
   f. discriminant = 37, two irrational roots, solutions are $x = 5, 6 + 4\sqrt{19}$

14. Sample answer: The discriminant is $b^2 - 4ac$ and originates from the radicand in the Quadratic Formula. When it is a perfect square, the radical disappears, yielding rational solutions. When it is positive but not a perfect square, then the radical remains, yielding irrational solutions. When the discriminant is negative, there is a negative under the radical, yielding complex solutions.

15. The discriminant is positive and a perfect square, so the quadratic equation has two rational roots.

16. Answers will vary, but the discriminant of the equation should be positive and not a perfect square. Sample answer: $x^2 - 4x + 2 = 0$. The discriminant is 8, which is positive and not a perfect square, so the quadratic equation has two irrational roots.

17. a. -192
   b. The discriminant is negative, which means that the solutions of the equation are not real. There are no real values of the time $t$ for which the height of the ball will reach 25 ft.

MATH TIP

In Item 17, remember to write the equation in standard form before you evaluate the discriminant.

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand how to use the discriminant to determine the nature of the solutions of a quadratic equation. To check for full conceptual understanding, pair students up and ask one student to explain the discriminant to the other student as if that student had been absent. The student playing the role of absentee can evaluate the explanation.
Solving $ax^2 + bx + c = 0$

### ACTIVITY 9 PRACTICE
Write your answers on notebook paper.
Show your work.

#### Lesson 9-1
For Items 1–8, solve each equation by taking the square root of both sides.

1. $4x^2 - 16 = 0$
2. $3x^2 = 54$
3. $9x^2 - 25 = 0$
4. $(x + 4)^2 - 9 = 0$
5. $3(x + 2)^2 = 15$
6. $-2(x - 4)^2 = 16$
7. $4(x - 8)^2 - 10 = 14$
8. $6(x + 3)^2 + 20 = 12$

9. Which of the following represents a formula that can be used to solve quadratic equations of the form $a(x - h)^2 + k = 0$, where $a \neq 0$?
   - A. $x = -h \pm \sqrt{\frac{k}{a}}$
   - B. $x = -h \pm \frac{k}{a}$
   - C. $x = h \pm \sqrt{-\frac{k}{a}}$
   - D. $x = h \pm \frac{k}{a}$

10. A plane begins flying due east from an airport at the same time as a helicopter begins flying due north from the airport. After half an hour, the plane and helicopter are 260 mi apart, and the plane is five times the distance from the airport as the helicopter.

   ![Diagram of a plane and a helicopter](image)

   a. Write an equation that can be used to determine $d$, the helicopter’s distance in miles from the airport after half an hour.
   b. Solve the equation and interpret the solutions.
   c. What are the average speeds of the plane and the helicopter? Explain.

#### Lesson 9-2
For Items 15–20, solve each equation by completing the square.

15. $x^2 + 2x = -3$
16. $x^2 + 10x = 11$
17. $x^2 + 3x - 4 = 0$
18. $x^2 - 5x = 0$
19. $x^2 - 8x = 0$
20. $x^2 + 6x + 5 = 0$

For Items 21–28, solve each equation by using the Quadratic Formula.

21. $x^2 + 12x + 16 = 0$
22. $3x^2 - 5x + 3 = 0$
23. $2x^2 + 6x = 25$
24. $4x^2 + 11x - 20 = 0$
25. $x^2 + 6x + 8 = 4x - 3$
26. $10x^2 - 5x = 9x + 8$
27. $4x^2 + x - 12 = 3x^2 - 5x$
28. $x^2 - 20x = 6x^2 - 2x + 20$

29. Write a formula that represents the solutions of a quadratic equation of the form $ax^2 + bx + c = 0$. Explain how you arrived at your formula.

30. Derive a formula for solving a quadratic equation of the form $x^2 + bx + c = 0$.

For Items 11–14, complete the square for each quadratic expression. Then factor the perfect square trinomial.

11. $x^2 + 10x + 25 = 0$
12. $x^2 - 16x + 64 = 0$
13. $x^2 + 3x + 4 = 0$
14. $x^2 - 3x + 1 = 0$

#### ACTIVITY 9 Continued

**ACTIVITY 9 PRACTICE**

1. $x = \pm 2\sqrt{2}$
2. $x = \pm 3\sqrt{5}$
3. $x = \pm \frac{4\sqrt{3}}{3}$
4. $x = -9, x = 1$
5. $x = -2, x = -\sqrt{5}$
6. $x = 4, x = 2\sqrt{2}$
7. $x = 8, x = 2\sqrt{6}$
8. $x = -3, x = \frac{2\sqrt{2}}{3}$
9. $c$
10. $d^2 + (5d)^2 = 260^2$ or equivalent

**ACTIVITY 9 PRACTICE**

11. $x^2 + 10x + 25 = 0$
12. $x^2 - 16x + 64 = 0$
13. $x^2 + 9x + \frac{81}{4} = 0$
14. $x^2 - x + \frac{1}{4} = 0$
15. $x = -1, x = 2i$
16. $x = 5\pm \sqrt{11}$
17. $x = -\frac{5}{2} \pm \frac{\sqrt{61}}{2}$
18. $x = -2, x = \frac{\sqrt{30}}{2}$
19. $x = \frac{5}{2} \pm \frac{\sqrt{365}}{6}$
20. $x = -\frac{4}{3} \pm \frac{\sqrt{10}}{6}$
21. $x = -6\pm \sqrt{30}$
22. $x = \frac{5}{2} \pm \frac{\sqrt{11}}{6}$
23. $x = \frac{-3\pm \sqrt{59}}{2}$
24. $x = \frac{5}{6}, x = 4, x = 7$
25. $x = -1 \pm i\sqrt{10}$
26. $x = \frac{2\pm \sqrt{129}}{10}$
27. $x = -3\pm \sqrt{21}$
28. $x = -9 \pm i\sqrt{19}$
29. $x = \frac{-n \pm \sqrt{n^2 - 4mp}}{2m}$ Use the Quadratic Formula with $a = m$, $b = n$, and $c = p$. 

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**Activity 9** Solving $ax^2 + bx + c = 0$ 149
31–36. Sample answers are given.

31. \( x = 2, x = -8 \); taking the square root of both sides; When you add 25 to both sides of the equation, each side is a perfect square.

32. \( x = \frac{9 \pm \sqrt{41}}{4} \); Quadratic Formula;
   The coefficient of the \( x^2 \)-term is not 1, which makes the other methods of solving more difficult.

33. \( x = -4, x = -3 \); factoring; Mental math shows that 12 has a factor pair of 4 and 3 with a sum of 7.

34. \( x = -\frac{9}{7}, x = 2 \); Quadratic Formula;
   The coefficient of the \( x^2 \)-term is not 1, which makes the other methods of solving more difficult.

35. \( x = -4 \pm \sqrt{53} \); completing the square; The variable terms are already isolated on one side, the coefficient of the \( x^2 \)-term is 1, and the coefficient of the \( x \)-term is even, all of which make completing the square easier.

36. \( x = \pm \sqrt{3}, x = \frac{3}{2} \); taking the square root of both sides; The equation has the form \( ax^2 + c = 0 \).

37. a. \( -2t^2 + 82t + 5 = 301 \) or equivalent
   \( t = 4, t = 37 \); Megan bought 4 tickets; the solution \( t = 37 \) can be excluded because customers may buy no more than 15 tickets.
   c. $75.25

38. \( -23 \), two complex roots
39. 0, one real (double) root
40. 529, two real, rational roots
41. 284, two real, irrational roots
42. 136, two real, irrational roots

43. D
44. a. \( a < 3 \) and \( a \neq 0 \)
   b. \( a = 3 \)
   c. \( a > 3 \)

45. a. \( -14s^2 + 440s - 2100 = 1200 \) or equivalent
   b. 8800
   c. Sample answer: The discriminant is positive, so the equation has two real solutions. However, if one or both values of \( s \) are negative, they would need to be excluded in this situation.

46. Sample answers:
   a. \( ax^2 + c = 0 \); taking the square root of both sides; You can solve the equation for \( x \) and then take the square root of both sides to solve for \( x \).
   b. \( ax^2 + bx = 0 \); factoring; You can factor \( x \) from the left side of the equation to get \( x(ax + b) = 0 \).
   c. \( x^2 + bx = -c \), where \( b \) is even; completing the square; The variable terms are already isolated on one side, the coefficient of the \( x^2 \)-term is 1, and the coefficient of the \( x \)-term is even, all of which make completing the square easier.
   d. \( x^2 + bx + c = 0 \), where \( c \) has a factor pair with a sum of \( b \); factoring; The coefficient of the \( x^2 \)-term is 1, and the information about \( b \) and \( c \) show that the equation is easy to factor.
   e. \( ax^2 + bx + c = 0 \), where \( a, b, \) and \( c \) are each greater than 10; Quadratic Formula; The coefficient of the \( x^2 \)-term is not 1 and the values of \( b \) and \( c \) are large, which makes the other methods of solving more difficult.

**MATHEMATICAL PRACTICES**

Look for and Use Make of Structure

46. Tell which method you would use to solve each quadratic equation having the given form. Then explain why you would use that method.
   a. \( ax^2 + c = 0 \)
   b. \( ax^2 + bx = 0 \)
   c. \( x^2 + bx = -c \), where \( b \) is even
   d. \( x^2 + bx + c = 0 \), where \( c \) has a factor pair with a sum of \( b \)
   e. \( ax^2 + bx + c = 0 \), where \( a, b, \) and \( c \) are each greater than 10
Applications of Quadratic Functions and Equations

NO HORSING AROUND

1. Kun-cha has 150 feet of fencing to make a corral for her horses. The barn will be one side of the partitioned rectangular enclosure, as shown in the diagram above. The graph illustrates the function that represents the area that could be enclosed.
   a. Write a function, \( A(x) \), that represents the area that can be enclosed by the corral.
   b. What information does the graph provide about the function?
   c. Which ordered pair indicates the maximum area possible for the corral? Explain what each coordinate tells about the problem.
   d. What values of \( x \) will give a total area of 1000 ft\(^2\) to 2000 ft\(^2\)?

2. Critique the reasoning of others. Tim is the punter for the Bitterroot Springs Mustangs football team. He wrote a function \( h(t) = 16t^2 + 8t + 1 \) that he thinks will give the height of a football in terms of \( t \), the number of seconds after he kicks the ball. Use two different methods to determine the values of \( t \) for which \( h(t) = 0 \). Show your work. Is Tim’s function correct? Why or why not?

3. Tim has been studying complex numbers and quadratic equations. His teacher, Mrs. Pinto, gave the class a quiz. Demonstrate your understanding of the material by responding to each item below.
   a. Write a quadratic equation that has two solutions, \( x = 2 + 5i \) and \( x = 2 - 5i \).
   b. Solve \( 3x^2 + 2x - 8 = 0 \), using an algebraic method.
   c. Rewrite \( \frac{4 + i}{3 - 2i} \) in the form \( a + bi \), where \( a \) and \( b \) are rational numbers.

Common Core State Standards for Embedded Assessment 1

- **HSA-CED.A.1** Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear functions.
- **HSA-CED.A.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
- **HSF-IF.B.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.
- **HSF-IF.B.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

Embedded Assessment 1

**Assessment Focus**
- Quadratic functions
- Quadratic equations
- Discriminants
- Complex numbers

**Answer Key**

1. a. \( A(x) = x(150 - 3x) = -3x^2 + 150x \)
   b. Students may note:
      - the \( x \)-intercepts for the graph appear to be at \( x = 0 \) and \( x = 50 \)
      - the graph is a parabola that is symmetric about the line \( x = 25 \)
      - the maximum \( y \)-value appears to be between 1800 and 1900
   c. The point indicating the maximum area for the enclosure is (25, 1875), where 25 is the length in feet of each part of the enclosure that is perpendicular to the barn, and 1875 is the total area in square feet enclosed by the corral.
   d. For 1000 ft\(^2\) of area, \( x \approx 7.922 \) ft or 42.078 ft. No value of \( x \) gives 2000 ft\(^2\) of total area.

2. Tim’s equation is written incorrectly since the value of the function is only equal to 0 when \( t \) is a negative value. This indicates that the height of the ball is never 0 ft after the ball is kicked, which would mean that the ball would never return to the ground. Students can choose one of several methods.
   - A graph of the function shows that there is no positive value of the time for which the height of the ball is 0 ft. It also shows that for positive values of the time, the height of the ball keeps increasing without ever decreasing.
   - Letting \( h(t) = 0 \) and solving for \( t \) by using the Quadratic Formula or completing the square gives \( t = \frac{-8 \pm \sqrt{64 - 4 \cdot 1 \cdot 4}}{8} = -\frac{1}{4} \), which shows that there is no positive value of the time at which the ball hits the ground.

3. a. \( x^2 - 4x + 29 = 0 \) or equivalent
   b. \( x = \frac{3}{4}, x = -2 \)
   c. \( \frac{10}{13}, \frac{11}{13} \)

**Teacher to Teacher**

Students may or may not realize that Tim’s equation does not make sense as a model for the height of the football over time. Tim’s equation is for a parabola with a minimum at the vertex instead of a parabola with a maximum height at the vertex.
Embedded Assessment 1

Teacher to Teacher
You may wish to read through the scoring guide with students and discuss the differences in the expectations at each level. Check that students understand the terms used.

Unpacking Embedded Assessment 2
Once students have completed this Embedded Assessment, turn to Embedded Assessment 2 and unpack it with them. Use a graphic organizer to help students understand the concepts they will need to know to be successful on Embedded Assessment 2.

Scoring Guide

<table>
<thead>
<tr>
<th>Mathematics Knowledge and Thinking (Items 1c, 1d, 2, 3a-c)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Effective understanding of and accuracy in solving quadratic equations algebraically or graphically</td>
<td>• Adequate understanding of solving quadratic equations algebraically or graphically, leading to solutions that are usually correct</td>
<td>• Partial understanding of and some difficulty solving quadratic equations algebraically or graphically</td>
<td>• Inaccurate or incomplete understanding of solving quadratic equations algebraically or graphically</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations</td>
<td>• Largely correct understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations</td>
<td>• Little or no understanding of the key features of graphs of quadratic functions and the relationship between zeros and solutions to quadratic equations</td>
<td></td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of how to perform operations with complex numbers</td>
<td>• Largely correct understanding of how to perform operations with complex numbers</td>
<td>• Difficulty performing operations with complex numbers</td>
<td>• Little or no understanding of how to perform operations with complex numbers</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem Solving (Items 1c, 1d, 2)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• An appropriate and efficient strategy that results in a correct answer</td>
<td>• A strategy that may include unnecessary steps but results in a correct answer</td>
<td>• A strategy that results in some incorrect answers</td>
<td>• No clear strategy when solving problems</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>Mathematical Modeling / Representations (Item 1)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Effective understanding of how to write a quadratic equation or function from a verbal description, graph or diagram</td>
<td>• Adequate understanding of how to write a quadratic equation or function from a verbal description, graph or diagram</td>
<td>• Partial understanding of how to write a quadratic equation or function from a verbal description, graph or diagram</td>
<td>• Little or no understanding of how to write a quadratic equation or function from a verbal description, graph or diagram</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of how to interpret features of the graphs of quadratic functions and the solutions to quadratic equations</td>
<td>• Largely correct understanding of how to interpret features of the graphs of quadratic functions and the solutions to quadratic equations</td>
<td>• Some difficulty interpreting the features of graphs of quadratic functions and the solutions to quadratic equations</td>
<td>• Inaccurate or incomplete interpretation of the features of graphs of quadratic functions and the solutions to quadratic equations</td>
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</table>

<table>
<thead>
<tr>
<th>Reasoning and Communication (Items 1b, 1c, 2)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Precise use of appropriate math terms and language to relate equations and graphs of quadratic functions and their key features to a real-world scenario</td>
<td>• Adequate descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario</td>
<td>• Misleading or confusing descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario</td>
<td>• Incomplete or inaccurate descriptions to relate equations and graphs of quadratic functions and their key features to a real-world scenario</td>
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</tr>
<tr>
<td>• Clear and accurate use of mathematical work to justify or refute a claim</td>
<td>• Correct use of mathematical work to justify or refute a claim</td>
<td>• Partially correct use of mathematical work to justify or refute a claim</td>
<td>• Incorrect or incomplete use of mathematical work to justify or refute a claim</td>
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</tr>
</tbody>
</table>

Common Core State Standards for Embedded Assessment 1 (cont.)

- HSN-CN.A.1 Know there is a complex number $i$ such that $i^2 = -1$, and every complex number has the form $a + bi$ with $a$ and $b$ real.
- HSN-CN.A.2 Use the relation $i^2 = -1$ and the commutative, associative, and distributive properties to add, subtract, and multiply complex numbers.
- HSN-CN.C.7 Solve quadratic equations with real coefficients that have complex solutions.
- HSN-CN.C.8Extend polynomial identities to the complex numbers.
Writing Quadratic Equations
What Goes Up Must Come Down
Lesson 10-1 Parabolas and Quadratic Equations

Learning Targets:
• Derive a general equation for a parabola based on the definition of a parabola.
• Write the equation of a parabola given a graph and key features.

SUGGESTED LEARNING STRATEGIES: Predict and Confirm, Discussion Groups, Interactive Word Wall, Create Representations, Close Reading

Take a look at the graphs shown below.

1. Make use of structure. Match each equation with one of the graphs above.

\[ x = \frac{1}{4}(y - 2)^2 - 1 \]  \[ y = \frac{1}{4}(x - 2)^2 - 1 \]

C  A

\[ y = -\frac{1}{4}(x - 2)^2 - 1 \]  \[ x = -\frac{1}{4}(y - 2)^2 - 1 \]

B  D

Activity Standards Focus
In Activity 10, students write equations of parabolas given a graph or key features of the parabola. They determine a quadratic function given three points on a plane that the function passes through. They also find a quadratic model for a given set of data values and use the model to make predictions about the data. Throughout this activity, emphasize the definition of a parabola and how the equation of a parabola relates to a quadratic function.

Lesson 10-1

PLAN

Pacing: 1 class period
Chunking the Lesson
#1–3  #4–6
Check Your Understanding
#10–11  #12–13  #14–17
Check Your Understanding
#21
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity
Have students make a table of values for the equations \( y = 2x, y = 2x^2, \) and \( y = 2x^3 \) using domain values of \(-3, -2, -1, 0, 1, 2, 3\). Then have them graph the equations.

1–3 Think-Pair-Share, Critique Reasoning Students will likely use several different methods to match the graphs to the equations. After students have shared their methods, ask students to determine the most efficient method.

Common Core State Standards for Activity 10

HSA-CED.A.1 Create equations and inequalities in one variable and use them to solve problems.
HSA-CED.A.2 Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
ACTIVITY 10

4–6 Activating Prior Knowledge, Graphic Organizer

When comparing the graphs of A and B to the graphs of C and D, students may note that A and B are functions while C and D are not. While this is true, it is important that students note that the orientations of the axes of symmetry are different. The notion of functions will be covered in Item 8.

Technology Tip

Point out that splitting an equation that includes the ± symbol results in two separate equations that are reflections of each other. Have students enter each equation separately. For additional technology resources, visit SpringBoard Digital.

Parabolas are defined both geometrically and algebraically. Geometrically, a parabola is a conic section and meets the geometric criteria set forth in this activity. Algebraically, a parabola is the graph of any quadratic equation.

Lesson 10-1
Parabolas and Quadratic Equations

2. Explain how you matched each equation with one of the graphs. Sample answer: A is the only graph that includes (4, 0), B is the only graph that includes (0, −2), C is the only graph that includes (0, 4), and D is the only graph that includes (−2, 0). Use substitution to determine which of these ordered pairs is a solution of each equation.

3. Use appropriate tools strategically. Use a graphing calculator to confirm your answers to Item 1. Which equations must be rewritten to enter them in the calculator? Rewrite any equations from Item 1 as necessary so that you can use them with your calculator.

Sample answer: rewrite \( x = \frac{1}{4} (y - 2)^2 - 1 \) as \( y = \frac{1}{4} x^2 + 2 \).

Check students’ calculator graphs.

4. a. How do graphs A and B differ from graphs C and D?

Sample answer: A and B are parabolas that open up or down. They are symmetric about a vertical line. C and D are parabolas that open right or left. They are symmetric about a horizontal line.

b. How do the equations of graphs A and B differ from the equations of graphs C and D?

Sample answer: The equations for A and B are solved for \( y \), and the expression equal to \( y \) is a quadratic expression in terms of \( x \). The equations for C and D are solved for \( x \), and the expression equal to \( x \) is a quadratic expression in terms of \( y \).
Lesson 10-1
Parabolas and Quadratic Equations

5. Work with your group. Consider graphs A and B and their equations.
   a. Describe the relationship between the graphs.
      Sample answer: The graphs are reflections of each other across the line $y = -1$.

   b. What part of the equation determines whether the graph opens up or down? How do you know?
      The equations are identical except for the sign of $\frac{1}{4}$, so the sign of this number determines whether the graph opens up or down. If the sign is positive, the graph opens up; if it is negative, the graph opens down.

   c. Attend to precision. What are the coordinates of the lowest point on graph A? What are the coordinates of the highest point on graph B? How do the coordinates of these points relate to the equations of the graphs?
      A: $(2, -1)$; B: $(2, -1)$; The $x$-coordinate is the number subtracted from $x$ inside the parentheses. The $y$-coordinate is the number added outside the parentheses.

6. Continue to work with your group. Consider graphs C and D and their equations.
   a. Describe the relationship between the graphs.
      Sample answer: The graphs are reflections of each other across the line $x = -1$.

   b. What part of the equation determines whether the graph opens to the right or left? How do you know?
      The equations are identical except for the sign of $\frac{1}{4}$, so the sign of this number determines whether the graph opens to the right or left. If the sign is positive, the graph opens to the right; if it is negative, the graph opens to the left.

   c. What are the coordinates of the leftmost point on graph C? What are the coordinates of the rightmost point on graph D? How do the coordinates of these points relate to the equations of the graphs?
      C: $(-1, 2)$; D: $(-1, 2)$; The $x$-coordinate is the number added outside the parentheses. The $y$-coordinate is the number subtracted from $y$ inside the parentheses.
Lesson 10-1
Parabolas and Quadratic Equations

The graphs shown at the beginning of this lesson are all parabolas. A parabola can be defined as the set of points that are equidistant from a fixed point and a fixed line. The fixed point is called the focus. The fixed line is called the directrix.

**Math Terms**

A parabola is the set of points in a plane that are equidistant from a fixed point and a fixed line. The fixed point is called the focus. The fixed line is called the directrix.

**Math Tip**

The distance between two points \((x_1, y_1)\) and \((x_2, y_2)\) is given by 
\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.
\]

**Math Tip**

The distance between a point and a horizontal line is the length of the vertical segment with one endpoint at the point and one endpoint on the line.

**Check Your Understanding**

7. Which equation does the graph at right represent? Explain your answer.
   A. \(y = -\frac{1}{2}(x + 2)^2 - 4\)
   B. \(y = -\frac{1}{2}(x + 2)^2 + 4\)
   C. \(y = -\frac{1}{2}(x - 2)^2 + 4\)
   Which of the equations in Item 1 represent functions? Explain your reasoning.

9. Consider the equation \(x = -2(y + 4)^2 - 1\). Without graphing the equation, tell which direction its graph opens. Explain your reasoning.

The graphs shown at the beginning of this lesson are all parabolas. A parabola can be defined as the set of points that are the same distance from a point called the focus and a line called the directrix.

10. The focus of graph A, shown below, is \((2, 0)\), and the directrix is the horizontal line \(y = -2\).

\[\text{Distance to focus: } \sqrt{(2 - (-2))^2 + (0 - 3)^2} = 5\]
Lesson 10-1
Parabolas and Quadratic Equations

b. Find the distance between the point \((-2, 3)\) and the directrix.
   distance to directrix: \(\sqrt{(-2 - (-2))^2 + (3 - 3)^2} = 5; x = 2\)

c. Reason quantitatively. Compare your answers in parts a and b. What do you notice?
The point \((-2, 3)\) on the parabola is the same distance from the focus as from the directrix.

11. The focus of graph D, shown below, is \((-2, 2)\), and the directrix is the vertical line \(x = 0\).

   a. The point \((-2, 4)\) is on the parabola. Show that this point is the same distance from the focus as from the directrix.
      distance to focus: \(\sqrt{(-2 - (-2))^2 + (4 - 2)^2} = 2\);
      distance to directrix: \(\sqrt{(0 - (-2))^2 + (4 - 4)^2} = 2\)

   b. The point \((-5, -2)\) is also on the parabola. Show that this point is the same distance from the focus as from the directrix.
      distance to focus: \(\sqrt{(-2 - (-5))^2 + (-2 - 2)^2} = 5\);
      distance to directrix: \(\sqrt{(0 - (-5))^2 + (-2 - (-2))^2} = 5\)

ACTIVITY 10

10–11 Create Representations, Identify a Subtask For students who struggle with either remembering or using the distance formula correctly, remind them that it is possible to determine the distance between two points that fall on a horizontal or vertical line by counting. Also, you can determine the distance between two points that do not fall on a vertical or horizontal line by constructing a right triangle and using the Pythagorean Theorem to determine the hypotenuse of the triangle.

Technology Tip
Use a dynamic mathematics software program such as GeoGebra to create a parabola by selecting a point and a directrix and using the parabola tool. You can then construct points on the parabola and measure the distances between the point and the focus and the point and the directrix.

For additional technology resources, visit SpringBoard Digital.
The focus of the parabola shown below is \((-2, -1)\), and the directrix is the line \(y = -5\).

12. a. Draw and label the **axis of symmetry** on the graph above. What is the equation of the axis of symmetry? 
   \[ x = -2 \]

   b. Explain how you identified the axis of symmetry of the parabola. 
   **Sample answer:** The directrix is horizontal, so I drew a vertical line through the focus.

13. a. Draw and label the **vertex** on the graph above. What are the coordinates of the vertex? 
   \((-2, -3)\)

   b. Explain how you identified the vertex of the parabola. 
   **Sample answer:** I drew a point where the axis of symmetry intersects the parabola.

   c. What is another way you could have identified the vertex? 
   **Sample answer:** I could have drawn a vertical segment from the focus to the directrix. Then I could have drawn a point at the midpoint of this segment.

**MATH TERMS**

The **axis of symmetry** is a line that divides the parabola into two congruent halves. The axis of symmetry passes through the focus and is perpendicular to the directrix.

The **vertex** is the point on the parabola that lies on the axis of symmetry. The vertex is the midpoint of the segment connecting the focus and the directrix.

**Differentiating Instruction**

For students who need a challenge, present them with the following task: Sketch a line through the focus of a parabola that is perpendicular to the axis of symmetry. This line will intersect the parabola in two points, \(P_1\) and \(P_2\). Explain why the distance between \(P_1\) and \(P_2\) is double the distance between the focus and the directrix of the parabola.
Lesson 10-1
Parabolas and Quadratic Equations

You can use what you have learned about parabolas to derive a general equation for a parabola whose vertex is located at the origin. Start with a parabola that has a vertical axis of symmetry, a focus of \((0, p)\), and a directrix of \(y = -p\). Let \(P(x, y)\) represent any point on the parabola.

14. Write, but do not simplify, an expression for the distance from point \(P\) to the focus.
   \[\sqrt{(x - 0)^2 + (y - p)^2}\] or equivalent

15. Write, but do not simplify, an expression for the distance from point \(P\) to the directrix.
   \[\sqrt{(x - x)^2 + (y - (-p))^2}\] or equivalent

16. Make use of structure. Based on the definition of a parabola, the distance from point \(P\) to the focus is the same as the distance from point \(P\) to the directrix. Set your expressions from Items 14 and 15 equal to each other, and then solve for \(y\).
   \[
   \sqrt{(x - 0)^2 + (y - p)^2} = \sqrt{(x - x)^2 + (y - (-p))^2}
   \]
   \[
   (x - 0)^2 + (y - p)^2 = (x - x)^2 + (y - (-p))^2
   \]
   \[
   x^2 + (y - p)^2 = (y + p)^2
   \]
   \[
   x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2
   \]
   \[
   x^2 - 2py = 2py
   \]
   \[
   x^2 = 4py
   \]
   \[
   \frac{x^2}{4p} = y
   \]

Differentiating Instruction

Have volunteers share their answers to these items. Some students may choose to simplify the expressions under the radicals prior to squaring both sides. Watch for the common student error of improperly expanding a squared binomial.
ACTIVITY 10 Continued

Check Your Understanding
Debrief students’ answers to these items to ensure that they understand concepts related to the general equation of a parabola.

Answers
18. \( x = \frac{1}{4p} y^2 \). Sample derivation:
   
   distance from \( P \) to focus = distance from \( P \) to directrix
   \[
   \sqrt{(x - p)^2 + (y - 0)^2}
   \]
   \[
   = \sqrt{(x - (-p))^2 + (y - y)^2}
   \]
   \[
   (x - p)^2 + (y - 0)^2
   \]
   \[
   = (x - (-p))^2 + (y - y)^2
   \]
   \[
   (x - p)^2 + y^2 = (x + p)^2
   \]
   \[
   x^2 - 2px + p^2 + y^2 = x^2 + 2px + p^2
   \]
   \[
   -2px + y^2 = 2px
   \]
   \[
   y^2 = 4px
   \]
   \[
   \frac{1}{4p} y^2 = x
   \]

19. \( y = \frac{-1}{12} x^2 \); The vertex and the focus of the parabola are on the \( y \)-axis, so the \( y \)-axis is the axis of symmetry. The parabola has its vertex at the origin and a vertical axis of symmetry, so its equation has the form \( y = \frac{1}{4p} x^2 \), where \( p \) is the \( y \)-coordinate of the focus. The focus is \((0, -3)\), and the equation of the parabola is \( y = \frac{1}{4(-3)} x^2 = -\frac{1}{12} x^2 \).

20. a. \( y = 4 \); The directrix is vertical, so the axis of symmetry is a horizontal line through the focus. The focus has a \( y \)-coordinate of 4, so the axis of symmetry is the line \( y = 4 \).

b. \((1, 4)\); The vertex is the midpoint of the segment that connects the focus and the directrix. The endpoints of this segment have coordinates \((3, 4)\) and \((-1, 4)\), so the vertex has coordinates \((1, 4)\).

c. To the right. The axis of symmetry is horizontal and the focus is to the right of the directrix, so the parabola opens to the right.

MATH TIP
A parabola always opens toward the focus and away from the directrix.

Lesson 10-1
Parabolas and Quadratic Equations

17. What is the general equation for a parabola with its vertex at the origin, a focus of \((0, p)\), and a directrix of \( y = -p \)?
   \[ y = \frac{1}{4p} x^2 \] or equivalent

18. See the diagram at right. Derive the general equation of a parabola with its vertex at the origin, a horizontal axis of symmetry, a focus of \((p, 0)\), and a directrix of \( x = -p \). Solve the equation for \( x \).

19. Model with mathematics. The vertex of a parabola is at the origin and its axis of symmetry is \((0, -3)\). What is the equation of the parabola? Explain your reasoning.

20. A parabola has a focus of \((3, 4)\) and a directrix of \( x = -1 \). Answer each question about the parabola, and explain your reasoning.
   a. What is the axis of symmetry?
   b. What is the vertex?
   c. In which direction does the parabola open?

You can also write general equations for parabolas that do not have their vertex at the origin. You will derive these equations later in this activity.

<table>
<thead>
<tr>
<th>Vertical Axis of Symmetry</th>
<th>Horizontal Axis of Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vertex ((h, k))</td>
<td>((h, k))</td>
</tr>
<tr>
<td>Focus ((h, k + p))</td>
<td>((h + p, k))</td>
</tr>
<tr>
<td>Directrix horizontal line</td>
<td>(y = k - p)</td>
</tr>
<tr>
<td></td>
<td>vertical line (x = h - p)</td>
</tr>
<tr>
<td>Equation (y = \frac{1}{4p} (x - h)^2 + k)</td>
<td>(x = \frac{1}{4p} (y - k)^2 + h)</td>
</tr>
</tbody>
</table>
21. **Reason quantitatively.** Use the given information to write the equation of each parabola.

a. axis of symmetry: \( y = 0 \); vertex: \((0, 0)\); directrix: \( x = \frac{1}{2} \)
\[ x = -\frac{1}{2} y^2 \]

b. vertex: \((3, 4)\); focus: \((3, 6)\)
\[ y = \frac{1}{8} (x - 3)^2 + 4 \]

c. vertex: \((-2, 1)\); directrix: \( y = 4 \)
\[ y = -\frac{1}{12} (x + 2)^2 + 1 \]

d. focus: \((-4, 0)\); directrix: \( x = 4 \)
\[ x = -\frac{1}{16} y^2 \]

e. opens up; focus: \((5, 7)\); directrix: \( y = 3 \)
\[ y = \frac{1}{8} (x - 5)^2 + 5 \]
ACTIVITY 10

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to equations of parabolas.

Answers

23. No. Sample explanation: To write the equation of a parabola, you need to know the value of \( p \). To determine the value of \( p \) given the vertex, you would also need to know either the focus or the directrix of the parabola.

24. vertex: \((1, 2)\); axis of symmetry: \( y = 2 \); focus: \((3, 2)\); directrix: \( x = -1 \)

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 10-1 PRACTICE

25. B

26.

27. axis of symmetry: \( y = 2 \); vertex: \((-3, 2)\); opens to the right

28. \( y = -\frac{1}{2}x^2 \)

29. \( x = \frac{1}{16}y^2 \)

30. \( x = -\frac{1}{20}(y - 5)^2 \)

31. \( y = \frac{1}{12}(x - 3)^2 - 4 \)

32. \( x = \frac{1}{4}(y - 4)^2 - 2 \)

22. Sample derivation:

\[
\text{distance from } P \text{ to focus} = \text{distance from } P \text{ to directrix} = \sqrt{(x-h)^2 + (y-(k+p))^2} = \sqrt{(x-h)^2 + (y-(k-p))^2}
\]

\[
(x-h)^2 + (y-(k+p))^2 = (x-h)^2 + (y-(k-p))^2
\]

\[
(x-h)^2 + y^2 - 2k + p + y + (k+p)^2 = y^2 - 2k + y + (k-p)^2
\]

\[
(x-h)^2 + y^2 - 2ky - 2py + p^2 + 2p + p^2 = y^2 - 2ky + 2py + k^2 - 2pk + p^2
\]

\[
(x-h)^2 - 2py + p^2 = 2py - 2pk
\]

\[
(x-h)^2 + 4pk = 4py
\]

\[
\frac{1}{4p}(x-h)^2 + k = y
\]
Learning Targets:
• Explain why three points are needed to determine a parabola.
• Determine the quadratic function that passes through three given points on a plane.

SUGGESTED LEARNING STRATEGIES: Create Representations, Quickwrite, Questioning the Text, Create Representations, Identify a Subtask

Recall that if you are given any two points on the coordinate plane, you can write the equation of the line that passes through those points. The two points are said to determine the line because there is only one line that can be drawn through them.

Do two points on the coordinate plane determine a parabola? To answer this question, work through the following items.

1. Follow these steps to write the equation of a quadratic function whose graph passes through the points (2, 0) and (5, 0).
   a. Write a quadratic equation in standard form with the solutions \( x = 2 \) and \( x = 5 \).
      \[ x^2 - 7x + 10 = 0 \] or a nonzero multiple of this equation
   b. Replace 0 in your equation from part a with \( y \) to write the corresponding quadratic function.
      Answers may vary depending on the equation in part a. Sample answer: \( y = x^2 - 7x + 10 \)
   c. Use substitution to check that the points (2, 0) and (5, 0) lie on the function’s graph.
      \[
      \begin{array}{ccc}
      0 & ? & (2)^2 - 7(2) + 10 \\
      0 & ? & 4 - 14 + 10 \\
      0 & = & 0 \checkmark
      \end{array}
      \]
      \[
      \begin{array}{ccc}
      0 & ? & (5)^2 - 7(5) + 10 \\
      0 & ? & 25 - 35 + 10 \\
      0 & = & 0 \checkmark
      \end{array}
      \]

2. a. Use appropriate tools strategically. Graph your quadratic function from Item 1 on a graphing calculator.
   Check students’ work.

   b. On the same screen, graph the quadratic functions
   \[ y = 2x^2 - 14x + 20 \] and \[ y = -x^2 + 7x - 10 \].
   Check students’ work.
Lesson 10-2
Writing a Quadratic Function Given Three Points

c. Describe the graphs. Do all three parabolas pass through the points (2, 0) and (5, 0)?

Answers may vary, but students should note that all three parabolas pass through the points (2, 0) and (5, 0).

Sample answer: Two of the parabolas open upward, and one opens downward. One parabola is narrower than the others. However, all of the parabolas have the same x-intercepts: 2 and 5.


No. Sample explanation: My graph of the three parabolas shows that more than one parabola can be drawn through the same pair of points, (2, 0) and (5, 0). So, two points are not enough to determine a parabola.

Three points in the coordinate plane that are not on the same line determine a parabola given by a quadratic function. If you are given three noncollinear points on the coordinate plane, you can write the equation of the quadratic function whose graph passes through them.

Consider the quadratic function whose graph passes through the points (1, 2), (3, 0), and (5, 6).

4. Write an equation by substituting the coordinates of the point (1, 2) into the standard form of a quadratic function, \( y = ax^2 + bx + c \).

\[ 2 = a + b + c \text{ or equivalent} \]

5. Write a second equation by substituting the coordinates of the point (3, 0) into the standard form of a quadratic function.

\[ 0 = 9a + 3b + c \text{ or equivalent} \]

6. Write a third equation by substituting the coordinates of the point (5, 6) into the standard form of a quadratic function.

\[ 6 = 25a + 5b + c \text{ or equivalent} \]

MATH TIP

Three or more points are collinear if they lie on the same straight line.

Universal Access

Using algebraic methods to solve a system of equations in three variables can be time-consuming and frustrating to students. Consider allowing students to use matrix equations and their graphing calculators to determine the values of \( a \), \( b \), and \( c \). Doing this will keep the lesson focused on finding the equation of the parabola.
Lesson 10-2
Writing a Quadratic Function Given Three Points

7. Use your equations from Items 4–6 to write a system of three equations in the three variables \(a\), \(b\), and \(c\).
\[
\begin{align*}
a + b + c &= 2 \\
9a + 3b + c &= 0 \quad \text{or equivalent} \\
25a + 5b + c &= 6
\end{align*}
\]

8. Use substitution or Gaussian elimination to solve your system of equations for \(a\), \(b\), and \(c\).
\[a = 1, \ b = -5, \ c = 6\]

9. Now substitute the values of \(a\), \(b\), and \(c\) into the standard form of a quadratic function.
\[y = x^2 - 5x + 6\]

10. Model with mathematics. Graph the quadratic function to confirm that it passes through the points (1, 2), (3, 0), and (5, 6).
Lesson 10-2
Writing a Quadratic Function Given Three Points

Check Your Understanding

11. Describe how to write the equation of a quadratic function whose graph passes through three given points.

12. a. What happens when you try to write the equation of the quadratic function that passes through the points (0, 4), (2, 2), and (4, 0)?
   b. What does this result indicate about the three points?

13. a. Reason quantitatively. The graph of a quadratic function passes through the point (2, 0). The vertex of the graph is (−2, −16). Use symmetry to identify another point on the function’s graph. Explain how you determined your answer.
   b. Write the equation of the quadratic function.

Lesson 10-2 Practice

Write the equation of the quadratic function whose graph passes through each set of points.

14. (−3, 2), (−1, 0), (1, 6)
15. (−2, −5), (0, −3), (1, 4)
16. (−1, −5), (1, −9), (4, 0)
17. (−3, 7), (0, 4), (1, 15)
18. (1, 0), (2, −7), (5, −16)
19. (−2, −11), (−1, −12), (1, 16)
20. The table below shows the first few terms of a sequence. This sequence can be described by a quadratic function, where \( f(n) \) represents the \( n \)th term of the sequence. Write the quadratic function that describes the sequence.

<table>
<thead>
<tr>
<th>Term Number, ( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term of Sequence, ( f(n) )</td>
<td>2</td>
<td>6</td>
<td>12</td>
<td>20</td>
<td>30</td>
</tr>
</tbody>
</table>

21. A quadratic function \( A(s) \) gives the area in square units of a regular hexagon with a side length of \( s \) units.
   a. Use the data in the table below to write the equation of the quadratic function.
   b. Attend to precision. To the nearest square centimeter, what is the area of a regular hexagon with a side length of 8 cm?

<table>
<thead>
<tr>
<th>Side Length, ( s )</th>
<th>2</th>
<th>4</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area, ( A(s) )</td>
<td>( 6\sqrt{3} )</td>
<td>( 24\sqrt{3} )</td>
<td>( 54\sqrt{3} )</td>
</tr>
</tbody>
</table>

My Notes

Assess

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

Adapt

Check students’ answers to the Lesson Practice to ensure that they understand how to write the equation of a parabola in standard form given three points that lie on the parabola. For additional practice, students can make up their own problems. Have them select three points and write the equation of the parabola using the method learned in this activity. Show students how they can check their work using quadratic regression on their graphing calculators.

MATH TIP

A sequence is an ordered list of numbers or other items. Each number or item in a sequence is called a term.

CONNECT TO GEOMETRY

A regular hexagon is a six-sided polygon with all sides having the same length and all angles having the same measure.

11. Substitute the coordinates of each point into the standard form of a quadratic function, \( y = ax^2 + bx + c \). Write the 3 resulting equations as a system of equations. Then solve the system for the values of \( a \), \( b \), and \( c \). Finally, use the values of \( a \), \( b \), and \( c \) to write the equation of the quadratic function in standard form.

12. a. You find that \( a = 0 \), \( b = -2 \), and \( c = 4 \), which results in the function \( f(x) = -x + 4 \). This function is linear, not quadratic.
   b. The 3 points are on the same line, which means that you cannot write the equation of a quadratic function whose graph passes through the points.

13. a. \((-6, 0)\). Sample explanation: For a quadratic function, the axis of symmetry is a vertical line that passes through the vertex, so the axis of symmetry is \( x = -2 \). The point \((2, 0)\) is 4 units to the right of the axis of symmetry, so there will be another point on the graph of the function 4 units to the left of the axis of symmetry with the same \( y \)-coordinate. This point has coordinates \((-6, 0)\).
   b. \( y = x^2 + 4x - 12 \)
Lesson 10-3
Quadratic Regression

Learning Targets:
• Find a quadratic model for a given table of data.
• Use a quadratic model to make predictions.

SUGGESTED LEARNING STRATEGIES: Think Aloud, Discussion Groups, Create Representations, Interactive Word Wall, Quickwrite, Close Reading, Predict and Confirm, Look for a Pattern, Group Presentation

A model rocketry club placed an altimeter on one of its rockets. An altimeter measures the altitude, or height, of an object above the ground. The table shows the data the club members collected from the altimeter before it stopped transmitting a little over 9 seconds after launch.

Model Rocket Test

<table>
<thead>
<tr>
<th>Time Since Launch (s)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (m)</td>
<td>0</td>
<td>54</td>
<td>179</td>
<td>255</td>
<td>288</td>
<td>337</td>
<td>354</td>
<td>368</td>
<td>378</td>
<td>363</td>
</tr>
</tbody>
</table>

1. Predict the height of the rocket 12 seconds after launch. Explain how you made your prediction. Predictions and explanations will vary.

2. Model with mathematics. Make a scatter plot of the data on the coordinate grid below.

CONNECT TO PHYSICS
A model rocket is not powerful enough to escape Earth’s gravity. The maximum height that a model rocket will reach depends in part on the weight and shape of the rocket, the amount of force generated by the rocket motor, and the amount of fuel the motor contains.

MINI-LESSON: Second Differences

If students need additional help with how to use second differences to determine if a set of data is a good candidate for a quadratic model, a mini-lesson is available to provide practice.

See the Teacher Resources at SpringBoard Digital for a student page for this mini-lesson.
3. Enter the rocket data into a graphing calculator. Enter the time data as List 1 (L1) and the height data as List 2 (L2). Then use the calculator to perform a linear regression on the data. Write the equation of the linear model that results from the regression. Round coefficients and constants to the nearest tenth.

\[ y = 41.4x + 71.4 \]

4. Use a dashed line to graph the linear model from Item 3 on the coordinate grid showing the rocket data. See graph below Item 2.

5. a. **Attend to precision.** To the nearest meter, what height does the linear model predict for the rocket 12 seconds after it is launched? 568 m

   b. How does this prediction compare with the prediction you made in Item 1?
   
   Answers will vary.

6. **Construct viable arguments.** Do you think the linear model is a good model for the rocket data? Justify your answer.

   *Sample answer: No. The linear model indicates that the rocket was already about 71 m off the ground at the time it was launched, when its actual height at that time was 0 m. Also, the linear model indicates that the rocket’s height would continue to increase with time without the rocket ever landing. The actual data show that the rocket’s height is starting to decrease after 8 seconds.*

A linear regression is the process of finding a linear function that best fits a set of data. A quadratic regression is the process of finding a quadratic function that best fits a set of data. The steps for performing a quadratic regression on a graphing calculator are similar to those for performing a linear regression.
Lesson 10-3
Quadratic Regression

7. Use these steps to perform a quadratic regression for the rocket data.
   - Check that the data set is still entered as List 1 and List 2.
   - Press [STAT] to select the Statistics menu. Then move the cursor to highlight the Calculate (CALC) submenu.
   - Select 5:QuadReg to perform a quadratic regression on the data in Lists 1 and 2. Press [ENTER].
   - The calculator displays the values of $a$, $b$, and $c$ for the standard form of the quadratic function that best fits the data.

Write the equation of the quadratic model that results from the regression. Round coefficients and constants to the nearest tenth.

$y = -6.9x^2 + 103.8x - 11.8$

8. Graph the quadratic model from Item 7 on the coordinate grid showing the rocket data.
   See graph below Item 2.

9. Construct viable arguments. Contrast the graph of the linear model with the graph of the quadratic model. Which model is a better fit for the data?
   Sample answer: The quadratic model is a better fit for the data because the data points are closer to the parabola overall than to the line. Unlike the linear model, the quadratic model shows that the rocket will eventually return to ground level.

10. a. To the nearest meter, what height does the quadratic model predict for the rocket 12 seconds after it is launched?
    Predictions should be close to 240 m.

    b. How does this prediction compare with the prediction you made in Item 1?
    Answers will vary.

11. Reason quantitatively. Use the quadratic model to predict when the rocket will hit the ground. Explain how you determined your answer.
    Answers may vary but should be close to 15 s.
    Sample explanation: I set the height $y$ of the quadratic model equal to 0, and used the Quadratic Formula to solve for the time $x$. The solutions show that the rocket will hit the ground after about 14.9 s.
ACTIVITY 10  Continued

Check Your Understanding
Debrief students’ answers to these items to ensure that they understand concepts related to quadratic regression.

Answers

12. An underestimate; The parachute slows the rocket down, which means that it will take the rocket longer to reach the ground than the model predicts. The prediction from the quadratic model is an underestimate of the time at which the rocket will reach the ground.

13. a. Yes. Three noncollinear points determine a parabola, so you can perform a quadratic regression if you have at least 3 data points.

b. The model would fit the data set exactly because there is only 1 parabola representing a quadratic function that can be drawn through any set of 3 noncollinear points.

c. If the 3 points lie on the same line, the quadratic regression would show that the coefficient of the $x^2$-term is 0. In other words, the quadratic regression would result in a linear model. The linear model would fit the data exactly, because the 3 points lie on the same line.

LESSON 10-3 PRACTICE

Tell whether a linear model or a quadratic model is a better fit for each data set. Justify your answer, and give the equation of the better model.

14. 

<table>
<thead>
<tr>
<th>$x$</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
<th>18</th>
<th>20</th>
<th>22</th>
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<tbody>
<tr>
<td>$y$</td>
<td>19</td>
<td>15</td>
<td>13</td>
<td>11</td>
<td>9</td>
<td>9</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

15. 

<table>
<thead>
<tr>
<th>$x$</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
<th>12</th>
<th>14</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>10</td>
<td>22</td>
<td>26</td>
<td>35</td>
<td>45</td>
<td>50</td>
<td>64</td>
<td>66</td>
</tr>
</tbody>
</table>

The tables show time and height data for two other model rockets.

Rocket A

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (m)</td>
<td>0</td>
<td>54</td>
<td>179</td>
<td>255</td>
<td>288</td>
<td>337</td>
<td>354</td>
<td>368</td>
</tr>
</tbody>
</table>

Rocket B

<table>
<thead>
<tr>
<th>Time (s)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (m)</td>
<td>0</td>
<td>37</td>
<td>92</td>
<td>136</td>
<td>186</td>
<td>210</td>
<td>221</td>
<td>229</td>
</tr>
</tbody>
</table>

16. Use appropriate tools strategically. Use a graphing calculator to perform a quadratic regression for each data set. Write the equations of the quadratic models. Round coefficients and constants to the nearest tenth.

17. Use your models to predict which rocket had a greater maximum height. Explain how you made your prediction.

18. Use your models to predict which rocket hit the ground first and how much sooner. Explain how you made your prediction.

LEARNER-SUPPORTED ENGLISH LEARNER

Many students will require additional practice. Create additional data sets for students by writing a quadratic function in standard form and using it to provide approximate points that fall on the curve.

Complete quadratic regression. Also make sure that students understand the concept of regression such that they can justify an appropriate regression model given a set of data. Many students will require additional practice. Create additional data sets for students by writing a quadratic function in standard form and using it to provide approximate points that fall on the curve.

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ACTIVITY 10 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 10-1
Use the parabola shown in the graph for Items 1 and 2.

1. What is the equation of the parabola?
   A. $y = - (x - 1)^2 - 2$
   B. $y = - (x + 1)^2 - 2$
   C. $y = (x - 1)^2 - 2$
   D. $y = (x + 1)^2 + 2$

2. The focus of the parabola is $(-1, -\frac{9}{4})$, and the directrix is the line $y = -\frac{7}{4}$. Show that the point $(-2, -3)$ on the parabola is the same distance from the focus as from the directrix.

3. Graph the parabola given by the equation $x = \frac{1}{2}(y - 3)^2 + 3$.

4. Identify the following features of the parabola given by the equation $y = \frac{1}{8}(x - 4)^2 + 3$.
   a. vertex
   b. focus
   c. directrix
   d. axis of symmetry
   e. direction of opening

5. Describe the relationships among the vertex, focus, directrix, and axis of symmetry of a parabola.

6. The focus of a parabola is $(3, -2)$, and its directrix is the line $x = -5$. What are the vertex and the axis of symmetry of the parabola?

Lesson 10-2
Write the equation of the quadratic function whose graph passes through each set of points.

13. $(-3, 0), (-2, 3), (2, 5)$
14. $(-2, -6), (1, 0), (2, 10)$
15. $(-5, -3), (-4, 0), (0, -8)$
16. $(-3, 10), (-2, 0), (0, -2)$
17. $(1, 0), (4, 6), (7, -6)$
18. $(-2, -9), (-1, 0), (1, -12)$

12. Sample derivation:

   \[
   \text{distance from } P \text{ to focus} = \text{distance from } P \text{ to directrix}
   \]
   \[
   \sqrt{(x - (h + p))^2 + (y - k)^2} = \sqrt{(x - (h - p))^2 + (y - y)^2}
   \]
   \[
   (x - (h + p))^2 + (y - k)^2 = (x - (h - p))^2 + (y - y)^2
   \]
   \[
   x^2 - 2(h + p)x + h^2 + p^2 + (y - k)^2 = x^2 - 2(h - p)x + (h - p)^2
   \]
   \[
   x^2 - 2hx - 2px + h^2 + 2hp + p^2 + (y - k)^2 = x^2 - 2hx + 2px + h^2 - 2hp + p^2
   \]
   \[
   -2px + 2hp + (y - k)^2 = 2px - 2hp
   \]
   \[
   (y - k)^2 + 4hp = 4px
   \]
   \[
   \frac{1}{4p}(y - k)^2 + h = x
   \]
19. Answers may vary, but equations should be a nonzero multiple of $y = x^2 + 2x - 48$. Sample answer: The parabolas given by the equations $y = x^2 + 2x - 48$ and $y = -x^2 - 2x + 48$ both pass through the points $(-8, 0)$ and $(6, 0)$.

20. a. $(-1, 5)$. Sample explanation: For a quadratic function, the axis of symmetry is a vertical line that passes through the vertex, so the axis of symmetry is $x = 3$. The point $(7, 5)$ is 4 units to the right of the axis of symmetry, so there will be another point on the graph of the function 4 units to the left of the axis of symmetry with the same $y$-coordinate. This point has coordinates $(-1, 5)$.

b. $f(x) = \frac{1}{4}x^2 - \frac{3}{2}x + \frac{13}{4}$

21. Sample justification: A linear model is a better fit. The values of $y$ increase as $x$ increases without ever decreasing, which indicates the shape of a linear, not a quadratic, model. Linear model: $y = 4.9x + 18.2$

22. Sample justification: A quadratic model is a better fit. A graph of both models shows that the data points are closer to the quadratic model. Also, the values of $y$ first decrease and then begin to increase as $x$ increases, which indicates the shape of a quadratic, not a linear, model. Quadratic model: $y = 0.3x^2 - 5.0x + 23.8$

23. car: $y = 0.047x^2 + 2.207x + 0.214$; truck: $y = 0.064x^2 + 2.210x - 0.500$

24. Predictions should be close to 42 feet.

25. No. Sample explanation: Based on the quadratic model, the stopping distance for the truck at 60 mi/h is about 363 feet. This distance is greater than the distance between the truck and the intersection, so the driver will not be able to stop in time.

26. a. $y = -2.2x^2 + 454.9x - 12,637.0$

b. Yes. Sample explanation: A graph of the quadratic model and the data from the table shows that the graph of the model is close to the data points. Also, the monthly revenue increases and then decreases as the selling price increases, which indicates a quadratic model could be a good fit for the data.

c. Answers should be close to $103$.

**ADDITIONAL PRACTICE**

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

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**MATHEMATICAL PRACTICES**

**Use Appropriate Tools Strategically**

26. A shoe company tests different prices of a new type of athletic shoe at different stores. The table shows the relationship between the selling price and the monthly revenue per store the company made from selling the shoes.

<table>
<thead>
<tr>
<th>Selling Price ($)</th>
<th>Monthly Revenue per Store ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>9680</td>
</tr>
<tr>
<td>90</td>
<td>10,520</td>
</tr>
<tr>
<td>100</td>
<td>11,010</td>
</tr>
<tr>
<td>110</td>
<td>10,660</td>
</tr>
<tr>
<td>120</td>
<td>10,400</td>
</tr>
<tr>
<td>130</td>
<td>9380</td>
</tr>
</tbody>
</table>

a. Use a graphing calculator to determine the equation of a quadratic model that can be used to predict $y$, the monthly revenue per store in dollars when the selling price is $x$ dollars. Round values to the nearest tenth.

b. Is a quadratic model a good model for the data set? Explain.

c. Use your model to determine the price at which the company should sell the shoes to generate the greatest revenue.
### Lesson 11-1 Translations of Parabolas

#### Learning Targets:
- Describe translations of the parent function \( f(x) = x^2 \).
- Given a translation of the function \( f(x) = x^2 \), write the equation of the function.

**SUGGESTED LEARNING STRATEGIES:** Create Representations, Quickwrite, Group Presentation, Look for a Pattern, Discussion Groups

1. Graph the parent quadratic function, \( f(x) = x^2 \), on the coordinate grid below. Include the points that have \( x \)-values \(-2, -1, 0, 1, \) and \( 2 \) are key points that can be used when graphing any quadratic function as a transformation of the parent quadratic function.

![Parent Parabola Graph](image)

The points on the parent function graph that have \( x \)-values \(-2, -1, 0, 1, \) and \( 2 \) are key points that can be used when graphing any quadratic function as a transformation of the parent quadratic function.

2. Graph \( f(x) = x^2 \) on the coordinate grid below. Then graph and label \( g(x) = x^2 - 3 \) and \( h(x) = x^2 + 2 \).

![Graphs of \( f(x) = x^2 \), \( g(x) = x^2 - 3 \), and \( h(x) = x^2 + 2 \)](image)

3. Make use of structure. Identify and describe the transformations of the graph of \( f(x) = x^2 \) that result in the graphs of \( g(x) \) and \( h(x) \).

Sample answer: The transformations moved the graph vertically but did not change the shape.

**MATH TIP**
A parent function is the simplest function of a particular type. For example, the parent linear function is \( f(x) = x \). The parent absolute value function is \( f(x) = |x| \).

**MATH TIP**
A transformation of a graph of a parent function is a change in the position, size, or shape of the graph.

### Common Core State Standards for Activity 11

- **HSF-BF.B.3** Identify the effect on the graph of replacing \( f(x) \) by \( f(x) + k \), \( kf(x) \), \( f(kx) \), and \( f(x + k) \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.

- **HSF-IF.C.8** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.

### Activity Standards Focus
In Activity 11, students explore transformations of parabolas. Students also write quadratic functions in vertex form. Throughout this activity, emphasize the effects of coefficients and constants on the graphs of functions.

### PLAN

#### Pacing:
1 class period

#### Chunking the Lesson

- #1 #2–3 #4–5 #6
- Check Your Understanding
- #11–12
- Check Your Understanding
- Lesson Practice

### TEACH

#### Bell-Ringer Activity
Spend some time reviewing with students the following characteristics of linear equations:
- The parent function is \( y = x \). The slope is 1, and it has a \( y \)-intercept of zero.
- Changing the parent function to \( y = -x \) changes the slope from 1 to \(-1\).
- When a coefficient other than 1 precedes the \( x \)-term, \( y = mx \), it changes the slope or steepness of the line.
- The addition of a \( y \)-intercept, \( y = mx + b \), changes the point at which the line crosses the \( y \)-axis (vertical translation).

#### 1 Create Representations
Students will likely be familiar with the parent quadratic function, but emphasis in this item should be on the five key points that are listed.

#### 2-3 Create Representations, Quickwrite
Students may use graphing calculators to visualize the graphs. When drawing the graphs of these two functions, students must recognize that the \( y \)-coordinates of the key points are changing. This will help identify the transformations as vertical translations (or an equivalent verbal form that describes the vertical movement of the graph without changing the shape). Be certain that students are not just drawing a rough sketch.
4. Model with mathematics. Graph \( f(x) = x^2 \) on the coordinate grid below. Then graph and label \( g(x) = (x - 2)^2 \) and \( h(x) = (x + 3)^2 \).

5. Identify and describe the transformations of the graph of \( f(x) = x^2 \) that result in the graphs of \( g(x) \) and \( h(x) \).

Sample answer: The transformations moved the graph horizontally but did not change the shape.

6. Describe each function as a transformation of \( f(x) = x^2 \). Then use that information to graph each function on the coordinate grid.
   a. \( a(x) = (x - 1)^2 \) translated 1 unit right
   b. \( w(x) = x^2 + 4 \) translated 4 units up
Lesson 11-1
Translations of Parabolas

c. \( d(x) = (x + 3)^2 - 5 \) translated 3 units left and 5 units down

d. \( j(x) = (x - 1)^2 + 2 \) translated 1 unit right and 2 units up

Check Your Understanding

7. Express regularity in repeated reasoning. The graph of each function below is a translation of the graph of \( f(x) = x^2 \) by \( k \) units, where \( k > 0 \). For each function, tell which direction the graph of \( f(x) \) is translated.
   a. \( g(x) = x^2 + k \)
   b. \( h(x) = (x + k)^2 \)
   c. \( j(x) = x^2 - k \)
   d. \( m(x) = (x - k)^2 \)

8. What is the vertex of the function \( p(x) = x^2 - 5 \)? Justify your answer in terms of a translation of \( f(x) = x^2 \).

9. What is the axis of symmetry of the function \( q(x) = (x + 1)^2 \)? Justify your answer in terms of a translation of \( f(x) = x^2 \).

10. Reason abstractly. The function \( r(x) \) is a translation of the function \( f(x) = x^2 \). What can you conclude about the direction in which the parabola given by \( r(x) \) opens? Justify your answer.

Math Tip
If you need help with Item 7, try substituting a positive number for \( k \) and then graphing each function.

Answers

7. a. \( k \) units up
   b. \( k \) units to the left
   c. \( k \) units down
   d. \( k \) units to the right

8. \((0, -5)\). Sample explanation:
The graph of \( p(x) \) is the graph of \( f(x) = x^2 \) translated 5 units down. The vertex of \( f(x) \) is \((0, 0)\), so the vertex of \( p(x) \) will be 5 units down from \((0, 0)\) at \((0, -5)\).

9. \( x = -1 \). Sample explanation:
The graph of \( q(x) \) is the graph of \( f(x) = x^2 \) translated 1 unit to the left. The axis of symmetry of the graph of \( f(x) \) is the line \( x = 0 \), so the axis of symmetry of the graph of \( q(x) \) will be 1 unit to the left of the line \( x = 0 \) at \( x = -1 \).

10. The parabola given by \( r(x) \) opens upward. A translation shifts a graph to the left, to the right, up, and/or down but does not change the direction that the parabola opens. Because the parabola given by \( f(x) = x^2 \) opens upward, all translations of \( f(x) = x^2 \) will open upward (assuming no other transformations are performed).
Lesson 11-1
Translations of Parabolas

11. Each function graphed below is a translation of \( f(x) = x^2 \). Describe the transformation. Then write the equation of the transformed function.

(a) \( g(x) \)
\[ g(x) = x^2 - 4 \]
translated 4 units down,

(b) \( h(x) \)
\[ h(x) = (x + 2)^2 \]
translated 2 units left,

(c) \( j(x) \)
\[ j(x) = (x + 4)^2 - 1 \]
translated 4 units left and 1 unit down,

(d) \( k(x) \)
\[ k(x) = (x - 3)^2 + 3 \]
translated 3 units right and 3 units up,

If students are having trouble predicting how the graphs are going to translate, have them make a chart like the one shown to summarize what has been covered so far and to use as a source of reference.

<table>
<thead>
<tr>
<th>Translations of ( y = x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Vertical</strong></td>
</tr>
<tr>
<td>( y = x^2 + 2 )</td>
</tr>
<tr>
<td>( y = x^2 - 2 )</td>
</tr>
</tbody>
</table>
Lesson 11-1  
Translations of Parabolas

12. Use a graphing calculator to graph each of the equations you wrote in Item 11. Check that the graphs on the calculator match those shown in Item 11. Revise your answers to Item 11 as needed. Check students’ work.

Check Your Understanding

13. Explain how you determined the equation of \( k(x) \) in Item 11d.

14. Critique the reasoning of others. The graph shows a translation of \( f(x) = x^2 \). A student says that the equation of the transformed function is \( g(x) = (x - 4)^2 \). Is the student correct? Explain.

15. The graph of \( h(x) \) is a translation of the graph of \( f(x) = x^2 \). If the vertex of the graph of \( h(x) \) is \((-1, -2)\), what is the equation of \( h(x) \)? Explain your answer.

LESSON 11-1 PRACTICE

Make sense of problems. Describe each function as a transformation of \( f(x) = x^2 \).

16. \( g(x) = x^2 - 6 \)  
17. \( h(x) = (x + 5)^2 \)  
18. \( j(x) = (x - 2)^2 + 8 \)  
19. \( k(x) = (x + 6)^2 - 4 \)

Each function graphed below is a translation of \( f(x) = x^2 \). Describe the transformation. Then write the equation of the transformed function.

20.  
21.  

22. What is the vertex of the function \( p(x) = (x - 5)^2 + 4 \)? Justify your answer in terms of a translation of \( f(x) = x^2 \).

23. What is the axis of symmetry of the function \( q(x) = (x + 8)^2 - 10 \)? Justify your answer in terms of a translation of \( f(x) = x^2 \).

LESSON 11-1 PRACTICE

16. translation 6 units down  
17. translation 5 units left  
18. translation 2 units right and 8 units up  
19. translation 6 units left and 4 units down  
20. translation 1 unit down; \( m(x) = x^2 - 1 \)  
21. translation 2 units left and 1 unit up; \( n(x) = (x + 2)^2 + 1 \)

22. (5, 4). Sample explanation: The graph of \( p(x) \) is the graph of \( f(x) = x^2 \) translated 5 units right and 4 units up. The vertex of \( f(x) \) is \((0, 0)\), so the vertex of \( p(x) \) will be 5 units to the right and 4 units up from \((0, 0)\) at \((5, 4)\).

23. \( x = -8 \). Sample explanation: The graph of \( q(x) \) is the graph of \( f(x) = x^2 \) translated 8 units left and 10 units down. The axis of symmetry of \( f(x) \) is \( x = 0 \), so the axis of symmetry of \( q(x) \) will be 8 units to the left of \( x = 0 \) at \( x = -8 \).

ACTIVITY 11 Continued

Check Your Understanding

11. Check that the graphs on the calculator match those shown in Item 11. Revise your answers to Item 11 as needed. Check students’ work.

TECHNOLOGY TIP

When you graph a function on a graphing calculator, the distance between tick marks on the x-axis is not always the same as the distance between tick marks on the y-axis. To make these distances the same, press [WINDOW], and select 5: ZSquare. This step will make it easier to compare your calculator graphs to the graphs in Item 11.

ACTIVITY 11

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand writing equations of quadratic functions given information about a translation.

Answers

13. Sample answer: The vertex of the graph of \( k(x) = (x - 3, 3) \), which means that \( k(x) \) is a translation of \( f(x) \) by 3 units right and 3 units up. To show a translation 3 units right, subtract 3 from \( x \) inside the parentheses. To show a translation 3 units up, add 3 to the squared term. So, \( k(x) = (x - 3)^2 + 3 \).

14. No. Sample explanation: If the number subtracted from \( x \) is positive, the translation is to the right. The graph of \( g(x) \) is the graph of \( f(x) \) translated 4 units to the left, so to write the equation of \( g(x) \), you need to subtract -4 inside the parentheses. The correct equation is \( g(x) = (x - (-4))^2 \), which simplifies to \( g(x) = (x + 4)^2 \).

15. \( h(x) = (x + 1)^2 - 2 \). Sample explanation: The vertex of the graph of \( h(x) \) indicates that \( h(x) \) is a translation of \( f(x) \) by 1 unit to the left and 2 units down. To write the equation of \( h(x) \), subtract -1 from \( x \) inside the parentheses and add -2 to the squared term: \( h(x) = (x - (-1))^2 + (-2) = (x + 1)^2 - 2 \).

ASSESS

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand vertical and horizontal translations of the parent quadratic function \( y = x^2 \). Make sure that students can match equations to their graphs. Have students create a matching game using note cards. Each game should contain ten cards that have quadratic equations and ten cards that have the graphs of those equations. These sets of cards can be distributed to other students in the classroom as additional practice.
Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

Learning Targets:
• Describe transformations of the parent function $f(x) = x^2$.
• Given a transformation of the function $f(x) = x^2$, write the equation of the function.

SUGGESTED LEARNING STRATEGIES: Create Representations, Look for a Pattern, Group Presentation, Quickwrite, Identify a Subtask

1. Graph the function $f(x) = x^2$ as Y1 on a graphing calculator. Then graph each of the following functions as Y2. Describe the graph of each function as a transformation of the graph of $f(x) = x^2$.
   a. $g(x) = 2x^2$
   b. $h(x) = 4x^2$
   c. $j(x) = \frac{1}{2}x^2$
   d. $k(x) = \frac{1}{4}x^2$

2. Express regularity in repeated reasoning. Describe any patterns you observed in the graphs from Item 1.
   Sample answer: The graphs have the same vertex $(0, 0)$ and the same axis of symmetry $x = 0$. All the graphs open upward. The graphs are vertical stretches when the number multiplied by $x^2$ is greater than 1 and vertical shrinks when the number multiplied by $x^2$ is a fraction less than 1.

3. Graph the function $f(x) = x^2$ as Y1 on a graphing calculator. Then graph each of the following functions as Y2. Identify and describe the graph of each function as a transformation of the graph of $f(x) = x^2$.
   a. $g(x) = -x^2$
   a reflection over the x-axis
Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

b. \( h(x) = -4x^2 \)
   a reflection over the x-axis and a vertical stretch by a factor of 4

c. \( j(x) = -\frac{1}{4}x^2 \)
   a reflection over the x-axis and a vertical stretch by a factor of \( \frac{1}{4} \)

4. Describe any patterns you observed in the graphs from Item 3.
   Sample answer: The graphs have the same vertex \((0, 0)\) and the same axis of symmetry \((x = 0)\). All the graphs open downward. The graphs are vertical stretches when the number multiplied by \(x^2\) is less than \(-1\) and vertical shrinks when the number multiplied by \(x^2\) is between \(-1\) and 0.

5. Make a conjecture about how the sign of \(k\) affects the graph of \(g(x) = kx^2\) compared to the graph of \(f(x) = x^2\). Assume that \(k \neq 0\).
   If \(k\) is positive, the graph of \(g(x)\) is not reflected over the x-axis compared to the graph of \(f(x)\). If \(k\) is negative, the graph of \(g(x)\) is reflected over the x-axis compared to the graph of \(f(x)\).

6. Make a conjecture about whether the absolute value of \(k\) affects the graph of \(g(x) = kx^2\) when compared to the graph of \(f(x) = x^2\). Assume that \(k \neq 0\) and write your answer using absolute value notation.
   If \(|k| > 1\), the graph of \(g(x)\) is a vertical stretch of the graph of \(f(x)\) by a factor of \(|k|\). If \(|k| < 1\), the graph of \(g(x)\) is a vertical shrink of the graph of \(f(x)\) by a factor of \(|k|\).

7. Make use of structure. Without graphing, describe each function as a transformation of \(f(x) = x^2\).
   a. \( h(x) = 6x^2 \)
      a vertical stretch by a factor of 6
   b. \( j(x) = -\frac{1}{10}x^2 \)
      a vertical shrink by a factor of \(\frac{1}{10}\) and a reflection over the x-axis

3–6 Summarizing, Debriefing
Another way of summarizing these items is to explain to students that the value of \(k\) determines whether a parabola opens upward or downward (in other words, has a vertex that is a lowest point or a highest point). Additionally, the value of \(k\) determines the shape of the parabola as it compares to the parent function of \(y = x^2\). As \(|k|\) increases, the parabola stretches vertically, becoming narrower. As \(|k|\) decreases, the parabola shrinks vertically, becoming wider.

7 Predict and Confirm, Look for a Pattern
Students should be able to look back at the outcomes of Items 1–6 to predict (without graphing) what will result from changing the value of \(k\) in this item.
Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

11. Graph the function $f(x) = x^2$ as $Y1$ on a graphing calculator. Then graph each of the following functions as $Y2$. Identify and describe the graph of each function as a horizontal stretch or shrink of the graph of $f(x) = x^2$.

- a. $g(x) = (2x)^2$
- b. $h(x) = (4x)^2$
- c. $j(x) = \left(\frac{1}{2}x\right)^2$
- d. $k(x) = \left(\frac{1}{4}x\right)^2$

Check Your Understanding

8. The graph of $g(x)$ is a vertical shrink of the graph of $f(x) = x^2$ by a factor of $\frac{1}{6}$. What is the equation of $g(x)$?

9. Reason quantitatively. The graph of $h(x)$ is a vertical stretch of the graph of $f(x) = x^2$. If the graph of $h(x)$ passes through the point $(1, 7)$, what is the equation of $h(x)$? Explain your answer.

10. The graph of $j(x) = ks^2$ opens downward. Based on this information, what can you conclude about the value of $k$? Justify your conclusion.

**MATH TIP**

A horizontal stretch stretches a graph away from the $y$-axis by a factor and a vertical shrink shrinks the graph toward the $y$-axis by a factor.

- c. $p(x) = -9x^2$
  - a vertical stretch by a factor of 9 and a reflection over the $x$-axis
- d. $q(x) = \frac{1}{5}x^2$
  - a vertical shrink by a factor of $\frac{1}{5}$
Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

Work with your group on Items 12–16.

12. Describe any patterns you observed in the graphs from Item 11.
   Sample answer: The graphs have the same vertex (0, 0) and the same axis of symmetry (x = 0). All the graphs open upward. The graphs are horizontal shrink when the number multiplied by x inside the parentheses is greater than 1 and horizontal stretches when the number multiplied by x inside the parentheses is a fraction less than 1.

13. a. Use appropriate tools strategically. Graph the function \( f(x) = x^2 \) as Y1 on a graphing calculator. Then graph \( h(x) = (-x)^2 \) as Y2. Describe the result.
   The graph of \( h(x) \) is the same as the graph of \( f(x) \).

b. Reason abstractly. Explain why this result makes sense.
   Sample answer: The expression \((-x)^2\) is equal to \((-1 \cdot x)^2 = (-1)^2 \cdot x^2 = 1 \cdot x^2 = x^2\). So, the rule for \( h(x) \) is equivalent to the rule for \( f(x) \).

14. Make a conjecture about how the sign of \( k \) affects the graph of \( g(x) = (kx)^2 \) compared to the graph of \( f(x) = x^2 \). Assume that \( k \neq 0 \).
   The sign of \( k \) has no effect on the graph of \( g(x) \) compared to the graph of \( f(x) \).

15. Make a conjecture about whether the absolute value of \( k \) affects the graph of \( g(x) = (kx)^2 \) when compared to the graph of \( f(x) = x^2 \). Assume that \( k \neq 0 \).
   If \(|k| > 1\), the graph of \( g(x) \) is a horizontal shrink of the graph of \( f(x) \) by a factor of \( \frac{1}{|k|} \). If \(|k| < 1\), the graph of \( g(x) \) is a horizontal stretch of the graph of \( f(x) \) by a factor of \( \frac{1}{|k|} \). 

16. Describe each function as a transformation of \( f(x) = x^2 \).
   a. \( p(x) = (6x)^2 \)
      a horizontal shrink by a factor of \( \frac{1}{6} \)
   b. \( q(x) = \left(\frac{1}{10}x\right)^2 \)
      a horizontal stretch by a factor of 10

ACTIVITY 11 Continued

13–16 Summarizing, Debriefing
Emphasize that a horizontal shrink implies the parabola is narrower, and a horizontal stretch implies the parabola is wider. As addressed in the Debriefing for Items 11 and 12, demonstrate to students that when a graph is in the form \( g(x) = (kx)^2 \), the factor by which it shrinks or stretches horizontally is the reciprocal of \( k \), or \( \frac{1}{k} \). Furthermore, when \( k > 1 \), the graph is a horizontal shrink, and when \( k < 1 \), the graph is a horizontal stretch.

ELL Support
Since the terms shrink and stretch are referenced with frequency in this activity, it is beneficial to give ELL students an analogy of these concepts with something other than parabolas. Demonstrate stretching by pulling on a rubber band or a piece of elastic. Demonstrate shrinking by writing a letter on the board and writing a much smaller version of the same number beside it to show that it shrinks. Point out that some materials, when stretched in one direction, will shrink in the perpendicular direction. Ask students if they can think of anything in the real world that commonly shrinks or stretches. [Samples with laundry/clothing: If you wash wool, it will shrink. If you wash certain clothing items in hot water, they may shrink. If you wear an article of clothing that is too small, it will stretch beyond its original size.]
Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to horizontal shrinks and stretches of the parent parabola.

**Answers**

17. Sample answer: The graph of \( g(x) = 4x^2 \) is a vertical stretch of the graph of \( f(x) = x^2 \) by a factor of 4. Each point \((x, y)\) on the graph of \( f(x) \) maps to the point \((x, 4y)\) on the graph of \( g(x) \). The graph of \( h(x) = (4x)^2 \) is a horizontal shrink of the graph of \( f(x) = x^2 \) by a factor of \( \frac{1}{4} \). Each point \((x, y)\) on the graph of \( f(x) \) maps to the point \((\frac{1}{4}x, y)\) on the graph of \( h(x) \).

18. \( g(x) = \left(\frac{1}{2}x\right)^2 \)

19. \( h(x) = (5x)^2 \). Sample explanation: The graph of \( f(x) \) passes through the point \((5, 25)\). This point is shrunk toward the \( y \)-axis by a factor of \( \frac{1}{5} \) to the corresponding point \((1, 25)\) on the graph of \( h(x) \). The graph of \( h(x) \) is a horizontal shrink of the graph of \( f(x) \) by a factor of \( \frac{1}{5} \).

20. Each function graphed below is a transformation of \( f(x) = x^2 \). Describe the transformation. Then write the equation of the transformed function.

- **a.** Vertically stretched by a factor of 3, \( g(x) = 3x^2 \)
- **b.** Horizontally stretched by a factor of 6, \( h(x) = \left(\frac{1}{6}x\right)^2 \) (or vertically shrunk by a factor of \( \frac{1}{36} \), \( h(x) = \frac{1}{36}x^2 \))
- **c.** Vertically shrunk by a factor of \( \frac{1}{2} \) and reflected over the \( x \)-axis, \( j(x) = -\frac{1}{2}x^2 \)
Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

21. **Model with mathematics.** Multiple transformations can be represented in the same function. Describe the transformations from the parent function. Then graph the function, using your knowledge of transformations only.

a. \( f(x) = -4(x + 3)^2 + 2 \)
   - translate 3 left, reflect over x-axis, stretch vertically by factor of 4, translate 2 up

b. \( f(x) = 2(x - 4)^2 - 3 \)
   - translate 4 right, stretch vertically by factor of 2, translate 3 down

d. horizontally shrunk by a factor of \( \frac{1}{3} \), \( k(x) = (3x)^2 \) (or vertically stretched by a factor of 9, \( k(x) = 9x^2 \))

**MATH TIP**

When graphing multiple transformations of quadratic functions, follow this order:

1. horizontal translation
2. horizontal shrink or stretch
3. reflection over the x-axis and/or vertical shrink or stretch
4. vertical translation

**ACTIVITY 11 Continued**

**ACTIVITY 11**

21 Identify a Subtask, Create Representations, Group Presentation, Debriefing

All of the transformations introduced so far in this activity are combined in this culminating item. Students may correctly graph these transformations in different orders. For instance, from an order of operations approach, if students first complete any transformations that involve multiplication (stretch/shrink or reflect), then apply any transformations that involve addition/subtraction (translations), they will have an accurate graph. Conversely, if they view the translations as “moving the origin,” then apply the reflection and stretch/shrink transformations, they will also have a correct graph.

Careful monitoring while students are working and class debriefing after a group presentation are essential after this item.
Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

c. \( f(x) = 2(x + 1)^2 - 4 \)
   - translate 1 left, stretch vertically by a factor of 2, translate 4 down


d. \( f(x) = -(x - 3)^2 + 5 \)
   - translate 3 right, reflect over x-axis, translate 5 up

Check Your Understanding

22. Explain how you determined the equation of \( g(x) \) in Item 20a.

23. Without graphing, determine the vertex of the graph of \( h(x) = 2(x - 3)^2 + 4 \). Explain how you found your answer.

24. a. Start with the graph of \( f(x) = x^2 \). Reflect it over the x-axis and then translate it 1 unit down. Graph the result as the function \( p(x) \).
   - \( p(x) = -x^2 - 1 \) or equivalent

b. Start with the graph of \( f(x) = x^2 \). Translate it 1 unit down and then reflect it over the x-axis. Graph the result as the function \( q(x) \).
   - \( q(x) = -x^2 + 1 \) or equivalent

c. Yes; the graph of \( p(x) \) is different from the graph of \( q(x) \), so the order in which the transformations are performed matters.

   d. \( p(x) = -x^2 - 1 \) or equivalent;
      \( q(x) = -x^2 + 1 \) or equivalent

Answers

22. Sample answer: The graph appears to be a vertical stretch of \( f(x) = x^2 \).
   - The graph of \( f(x) = x^2 \) passes through the point (1, 1). This point is stretched away from the x-axis by a factor of 3 to the corresponding point (1, 3) on the graph of \( g(x) \).
   - The graph of \( g(x) \) is a vertical stretch of the graph of \( f(x) \) by a factor of 3. Therefore, \( g(x) = 3x^2 \).

23. (3, 4). Sample explanation: The graph of \( h(x) \) is a translation 3 units to the right of the graph of \( f(x) \).
   - This translation moves the vertex from (0, 0) to (3, 0). The translation is followed by a vertical stretch by a factor of 2. This stretch does not change the position of the vertex.
   - The stretch is then followed by a translation 4 units up. This translation moves the vertex from (3, 0) to (3, 4).

C. Yes; the graph of \( p(x) \) is different from the graph of \( q(x) \), so the order in which the transformations are performed matters.

   d. \( p(x) = -x^2 - 1 \) or equivalent;
      \( q(x) = -x^2 + 1 \) or equivalent
Lesson 11-2
Shrinking, Stretching, and Reflecting Parabolas

LESSON 11-2 PRACTICE
Describe each function as a transformation of \( f(x) = x^2 \).

25. \( g(x) = -5x^2 \)

26. \( h(x) = (8x)^2 \)

27. Make sense of problems. The graph of \( j(x) \) is a horizontal stretch of the graph of \( f(x) = x^2 \) by a factor of 7. What is the equation of \( j(x) \)?

Each function graphed below is a transformation of \( f(x) = x^2 \). Describe the transformation. Then write the equation of the transformed function.

28. \( g(x) = -5x^2 \)

29. \( h(x) = (8x)^2 \)

Describe the transformations from the parent function. Then graph the function, using your knowledge of transformations only.

30. \( n(x) = -3(x - 4)^2 \)

31. \( p(x) = \frac{1}{2}(x + 3) - 5 \)

31. Translated 3 units left, vertically shrunk by a factor of \( \frac{1}{2} \) and translated 5 units down.

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning.

See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 11-2 PRACTICE

25. Reflected over the \( x \)-axis and vertically stretched by a factor of 5

26. Horizontal shrink by a factor of \( \frac{1}{8} \)

27. \( j(x) = \left( \frac{x}{4} \right)^2 \)

28. Vertical shrink by a factor of \( \frac{1}{3} \);
   \( k(x) = \frac{1}{3}x^2 \)

29. Horizontal stretch by a factor of 9;
   \( m(x) = \left( \frac{x}{9} \right)^2 \) (or vertical shrink by a factor of 81; \( m(x) = \frac{1}{81}x^2 \))

30. Translated 4 units right, reflected over the \( x \)-axis and vertically stretched by a factor of 3

ADAPT

Check students’ answers to the Lesson Practice to ensure that they understand reflections over both axes and vertical and horizontal shrinks and stretches of the parent function \( y = x^2 \). Make sure that students can match equations to their graphs. Once all transformations have been covered, students may benefit from creating a graphic organizer of transformations and their effects on the graph. Students will need to apply these transformations to different parent functions throughout future lessons.
ACTIVITY 11

Lesson 11-3

PLAN

Pacing: 1 class period

Chucking the Lesson
Example A #1–2
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity
Have students complete the square in the following expressions. Reviewing this concept will help prepare them for writing quadratic equations in vertex form.

1. \(x^2 + 8x + \underline{\phantom{00}}\) \[16\]
2. \(y^2 + 5y + \underline{\phantom{00}}\) \[\frac{25}{4}\]
3. \(a^2 - \frac{1}{2}a + \underline{\phantom{00}}\) \[\frac{1}{16}\]

Developing Math Language
Just as a linear equation has a standard form of \(Ax + By = C\), the standard form of a quadratic equation is \(y = ax^2 + bx + c\). However, when graphing linear equations, mathematicians usually use the slope-intercept form of the equation because it is easy to graph the \(y\)-intercept and plot a second point from it using the slope. Once you have two points, you can draw a line. When graphing quadratic equations, you usually use the vertex form of the equation, \(y = a(x - h)^2 + k\), because you can easily discern and plot the vertex \((h, k)\). The horizontal translation of the parabola can be determined from the value of \(h\), and the vertical translation of the parabola can be determined from the value of \(k\). Additionally, the value of \(a\) determines both the direction and shape of the parabola.

Example A Activating Prior Knowledge, Debriefing, Create a Plan
There are a few subtle differences between the steps of Example A and the steps you followed when completing the square to solve quadratic equations. The reason for these differences is due to the fact that students should only be able to work on one side of the equation. Students should group the \(ax^2 + bx\) terms and factor out \(a\), leaving \(a\left(x^2 + \frac{b}{a}x\right)\). Now complete the square as before.

Note that the value \(-\frac{b^2}{4a}\), which is added inside the parentheses after the term \(\frac{b}{a}x\), when multiplied by the \(a\) preceding the parentheses, gives the product \(-\frac{b^2}{4a}\).

Example A (continued) This value is subtracted from \(c\) outside the parentheses in order to maintain equality with the original function. This sum of \(-\frac{b^2}{4a}\) and \(c\) will determine the value of \(k\). Once you have completed the square, you factor the perfect square trinomial just like before. You should now have an equation in vertex form.

Learning Targets:
- Write a quadratic function in vertex form.
- Use transformations to graph a quadratic function in vertex form.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Marking the Text, Create Representations, Group Presentation, RAFT

A quadratic function in standard form, \(f(x) = ax^2 + bx + c\), can be changed into vertex form by completing the square.

Example A

Write \(f(x) = 3x^2 - 12x + 7\) in vertex form.

Solution:

Step 1: Factor the leading coefficient from the quadratic and linear terms.

\[f(x) = 3(x^2 - 4x + \underline{\phantom{00}}) + 7\]

Step 2: Complete the square by taking half the linear coefficient \((\frac{b}{2} = \frac{-4}{2}) = -2\). Squaring it \(((-2)^2 = 4)\), and then adding it inside the parentheses.

\[f(x) = 3\left(x^2 - 4x + 4\right) + 7 - 3(4)\]

Step 3: To maintain the value of the expression, multiply the leading coefficient \(3\) by the number added inside the parentheses \(4\). Then subtract that product \((12)\).

\[f(x) = 3\left(x^2 - 4x + 4\right) - 3(4) + 7 - (12)\]

Step 4: Write the trinomial inside the parentheses as a perfect square. The function is in vertex form.

\[f(x) = \left(x - 2\right)^2 - 5\]

Solution: The vertex form of \(f(x) = 3x^2 - 12x + 7\) is \(f(x) = 3(x - 2)^2 - 5\).

Try These A

Make use of structure. Write each quadratic function in vertex form.

Show your work.

a. \(f(x) = 5x^2 + 40x - 3\) \(f(x) = 5(x + 4)^2 - 83\)

b. \(g(x) = -4x^2 - 12x + 1\) \(g(x) = -4\left(x + \frac{3}{2}\right)^2 + 10\)
Lesson 11-3
Vertex Form

1. Make sense of problems. Write each function in vertex form. Then describe the transformation(s) from the parent function and graph without the use of a graphing calculator.
   a. $f(x) = -2x^2 + 4x + 3$
      $f(x) = -2(x - 1)^2 + 5$; translate 1 right, reflect over the x-axis, vertical stretch by factor of 2, and translate 5 up.

   b. $g(x) = \frac{1}{2}x^2 + 3x + \frac{3}{2}$
      $g(x) = \frac{1}{2}(x + 3)^2 - 3$; translate 3 left, vertical shrink by factor of $\frac{1}{2}$, and translate 3 down.

2. Consider the function $f(x) = 2x^2 - 16x + 34$.
   a. Write the function in vertex form.
      $f(x) = 2(x - 4)^2 + 2$

   b. What is the vertex of the graph of the function? Explain your answer.
      (4, 2). Sample explanation: In the vertex form of the equation, the value of $h$ is 4 and the value of $k$ is 2.

3. Debrief students’ answers to these items to ensure that they understand concepts related to writing equations for parabolas and quadratic functions in vertex form.

   a. Sample answer:
      (1) Write the variable terms in parentheses:
         $f(x) = (x^2 + 6x) + 11$.
      (2) Next, decide what number to add inside the parentheses to complete the square. Divide the coefficient of the $x$-term by 2:
         $6 \div 2 = 3$. Then square the result: $3^2 = 9$. You need to add 9 to complete the square.
      (3) Add 9 inside the parentheses. To keep the expression on the right side of the equation balanced, subtract 9 outside the parentheses:
         $f(x) = (x^2 + 6x + 9) - 9 + 11$.
      (4) Factor the expression in parentheses:
         $f(x) = (x + 3)^2 - 9 + 11$.
      (5) Combine the constant terms:
         $f(x) = (x + 3)^2 + 2$.
         The equation of the function is now in vertex form, $f(x) = a(x - h)^2 + k$, with $a = 1$, $h = -3$ and $k = 2$. 

4. Sample answer: It is easier to determine the vertex of the graph of the function when the equation is written in vertex form. It is also easier to graph the function as a set of transformations of the parent function when the equation is written in vertex form.

5. Write 1 in the first box because adding 1 completes the square for the quadratic expression $x^2 - 2x$ inside the parentheses. Write 4 in the second box because subtracting 4 outside the parentheses keeps the expression on the right side of the equation balanced.
LESSON 11-3 PRACTICE

6. \( g(x) = (x + 3)^2 - 4 \); translated 3 units left and 4 units down

7. \( h(x) = (x - 4)^2 + 1 \); translated 4 units right and 1 unit up

8. \( f(x) = 2(x + 1)^2 + 3 \); translated 1 unit left, vertically stretched by a factor of 2 and translated 3 units up

9. \( k(x) = -3(x - 2)^2 + 5 \); translated 2 units right, reflected over the x-axis, vertically stretched by a factor of 3 and translated 5 units up

10. \( f(x) = (x - 10)^2 + 7 \); vertex: \((10, 7)\); axis of symmetry: \(x = 10\); opens upward

11. \( f(x) = -(x + 3)^2 + 3 \); vertex: \((-3, -3)\); axis of symmetry: \(x = -3\); opens downward

12. \( f(x) = 5(x - 2)^2 + 11 \); vertex: \((2, 11)\); axis of symmetry: \(x = 2\); opens upward

13. \( f(x) = -2(x + 3)^2 + 23 \); vertex: \((-3, 23)\); axis of symmetry: \(x = -3\); opens downward

14. Both are correct. The standard form of the equation is \( f(x) = 1x^2 + 0x + (-5) \), which simplifies to \( f(x) = x^2 - 5 \). The vertex form of the equation is \( f(x) = 1(x - 0)^2 + (-5) \), which also simplifies to \( f(x) = x^2 - 5 \).
ACTIVITY 11 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 11-1
For each function, identify all transformations of the function \( f(x) = x^2 \). Then graph the function.

1. \( g(x) = x^2 + 1 \)
2. \( g(x) = (x - 4)^2 \)
3. \( g(x) = (x + 2)^2 + 3 \)
4. \( g(x) = (x - 3)^2 - 4 \)

Each function graphed below is a translation of \( f(x) = x^2 \). Describe the transformation. Then write the equation of the transformed function.

5. 

\[ g(x) = \begin{cases} 
1 & \text{if } x \leq -2 \\
3 & \text{if } x > -2 
\end{cases} \]

6. 

\[ h(x) = \begin{cases} 
2 & \text{if } x \leq 3 \\
6 & \text{if } x > 3 
\end{cases} \]

Use transformations of the parent quadratic function to determine the vertex and axis of symmetry of the graph of each function.

7. \( g(x) = (x - 8)^2 \)
8. \( g(x) = (x + 6)^2 - 4 \)

9. Translate 6 units right
10. Translate 10 units down
11. Translate 9 units right and 6 units up
12. Translate 4 units left and 8 units down
13. The function \( g(x) \) is a translation of \( f(x) = x^2 \). The vertex of the graph of \( g(x) \) is \((-4, 7)\). What is the equation of \( g(x) \)? Explain your answer.

Lesson 11-2
For each function, identify all transformations of the function \( f(x) = x^2 \). Then graph the function.

14. \( g(x) = \frac{1}{3} x^2 \)
15. \( g(x) = \frac{1}{5} x^2 \)
16. \( g(x) = \frac{1}{4} (x - 3)^2 \)
17. \( g(x) = -2(x + 3)^2 + 1 \)
18. \( g(x) = -3(x + 2)^2 - 5 \)

Write a quadratic function \( g(x) \) that represents each transformation of the function \( f(x) = x^2 \).

19. Shrink horizontally by a factor of \( \frac{1}{4} \)
20. Stretch vertically by a factor of 8
21. Shrink vertically by a factor of \( \frac{1}{3} \), translate 6 units up
22. Translate 1 unit right, stretch vertically by a factor of \( \frac{3}{2} \), reflect over the \( x \)-axis, translate 7 units up

18. Translate 2 units left, reflect over the \( x \)-axis, stretch vertically by a factor of 3, translate 5 units down.

19. \( g(x) = (4x)^2 \)
20. \( g(x) = 8x^2 \)
21. \( g(x) = \frac{1}{3} x^2 + 6 \)
22. \( g(x) = -\frac{3}{2} (x - 1)^2 + 7 \)

16. Translate 3 units to the right, shrink vertically by a factor of \( \frac{1}{2} \).
17. Translate 3 units left, reflect over the \( x \)-axis, stretch vertically by a factor of 2, translate 1 unit up.
ACTIVITY 11 Continued

23. vertical stretch by a factor of 3 and reflect over the x-axis; \( g(x) = -3x^2 \)
24. horizontal stretch by a factor of 3; \( h(x) = \left(\frac{1}{3}x\right)^2 \) (or vertical shrink by factor of 9; \( h(x) = \frac{1}{9}x^2 \))
25. D
26. \( g(x) = (x - 2)^2 - 5 \); Translate 2 units right and 5 units down.

![Graph of a parabola](image)

27. \( g(x) = -2(x - 3)^2 + 1 \); Translate 3 units right, reflect over the x-axis, vertically stretch by a factor of 2 and translate 1 unit up.

![Graph of a parabola](image)

28. \( g(x) = 3(x + 1)^2 - 2 \); Translate 1 unit left, vertically stretch by a factor of 3 and translate 2 units down.

![Graph of a parabola](image)

29. \( f(x) = (x - 8)^2 + 7 \); vertex: (8, 7); axis of symmetry: \( x = 8 \); opens upward
30. \( f(x) = 2(x + 9)^2 - 20 \); vertex: (-9, -20); axis of symmetry: \( x = -9 \); opens upward
31. \( f(x) = -3(x - 1)^2 + 12 \); vertex: (1, 12); axis of symmetry: \( x = 1 \); opens downward
32. \( f(x) = (x - 1)^2 + 4 \); vertex: (1, 4); axis of symmetry: \( x = 1 \); opens upward

ADDITIONAL PRACTICE

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

ACTIVITY 11 continued

Each function graphed below is a transformation of \( f(x) = x^2 \). Describe the transformation. Then write the equation of the transformed function.

23. [Graph of a parabola]

24. [Graph of a parabola]

25. Which of these functions has the widest graph when they are graphed on the same coordinate plane?
   A. \( f(x) = -2x^2 \)
   B. \( f(x) = 5x^2 \)
   C. \( f(x) = \frac{1}{2}x^2 \)
   D. \( f(x) = -\frac{1}{2}x^2 \)

Lesson 11-3

Write each function in vertex form. Then describe the transformation(s) from the parent function and use the transformations to graph the function.

26. \( g(x) = x^2 - 4x - 1 \)
27. \( g(x) = -2x^2 + 12x - 17 \)
28. \( g(x) = 3x^2 + 6x + 1 \)

33. a. \( h(t) = -16\left(t - \frac{11}{16}\right)^2 + \frac{185}{16} \)
   b. 12 ft. Sample explanation: The vertex form shows that the graph of the function opens downward and its vertex is \( \left(\frac{11}{16}, \frac{185}{16}\right) \). The maximum value of the function is \( \frac{185}{16} = 11.5 \), or about 12 ft.
   c. 0.7 s. Sample explanation: The maximum value of the function is \( \left(\frac{11}{16}, \frac{185}{16}\right) \). The maximum height of \( \frac{185}{16} \) ft occurs when \( t = \frac{11}{16} \) s, or about 0.7 s.

34. B

35. Yes, the argument is valid. The graph of \( g(x) \) is a reflection of \( f(x) = x^2 \) over the x-axis followed by a translation 5 units down. The reflection over the x-axis results in the graph of \( g(x) \) opening downward, which means that \( g(x) \) has a maximum value at its vertex. The vertex form of the equation is \( g(x) = -\left(x - 0\right)^2 - 5 \), confirming that the vertex is \((0, -5)\). The greatest value of \( g(x) \) is -5, which means that there is no real value of \( x \) for which \( g(x) = 0 \).
A zoo is constructing a new exhibit of African animals called the Safari Experience. A path called the Lion Loop will run through the exhibit. The Lion Loop will have the shape of a parabola and will pass through these points shown on the map: (3, 8) near the lions, (7, 12) near the hyenas, and (10, 4.5) near the elephants.

1. Write the standard form of the quadratic function that passes through the points (3, 8), (7, 12), and (10, 4.5). This function models the Lion Loop on the map.

2. A lemonade stand will be positioned at the vertex of the parabola formed by the Lion Loop.
   a. Write the equation that models the Lion Loop in vertex form, \( y = a(x - h)^2 + k \).
   b. What are the map coordinates of the lemonade stand? Explain how you know.

3. A graphic artist needs to draw the Lion Loop on the map.
   a. Provide instructions for the artist that describe the shape of the Lion Loop as a set of transformations of the graph of \( f(x) = x^2 \).
   b. Use the transformations of \( f(x) \) to draw the Lion Loop on the map.

4. The Safari Experience will also have a second path called the Cheetah Curve. This path will also be in the shape of a parabola. It will open to the right and have its focus at the cheetah exhibit at map coordinates (5, 6).
   a. Choose a vertex for the Cheetah Curve. Explain why the coordinates you chose for the vertex are appropriate.
   b. Use the focus and the vertex to write the equation that models the Cheetah Curve.
   c. What are the directrix and the axis of symmetry of the parabola that models the Cheetah Curve?
   d. Draw and label the Cheetah Curve on the map.

**Common Core State Standards for Embedded Assessment 2**

- **HSA-CED.A.1** Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear functions.
- **HSA-CED.A.2** Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
- **HSA-IF.C.8** Write a function defined by an expression in different but equivalent forms to reveal and explain different properties of the function.
- **HSF-BF.B.3** Identify the effect on the graph of replacing \( f(x) \) by \( f(x) + k \), \( kf(x) \), \( f(kx) \), and \( f(x + k) \) for specific values of \( k \) (both positive and negative); find the value of \( k \) given the graphs. Experiment with cases and illustrate an explanation of the effects on the graph using technology.
Unpacking Embedded Assessment 3

Once students have completed this Embedded Assessment, turn to Embedded Assessment 3 and unpack it with them. Use a graphic organizer to help students understand the concepts they will need to know to be successful on Embedded Assessment 3.

### Embedded Assessment 2

**Use after Activity 11**

1. Use the graphic organizer for Embedded Assessment 2 and unpack it.
2. **Embedded Assessment 3**
   - Use after Activity 11

### Scoring Guide

<table>
<thead>
<tr>
<th>Mathematics Knowledge and Thinking (Items 1, 2a, 4a-c)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
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<tbody>
<tr>
<td>• Effective understanding of quadratic functions as transformations of ( f(x) = x^2 )</td>
<td>• Adequate understanding of quadratic functions as transformations of ( f(x) = x^2 )</td>
<td>• Partial understanding of quadratic functions as transformations of ( f(x) = x^2 )</td>
<td>• Inaccurate or incomplete understanding of quadratic functions as transformations of ( f(x) = x^2 )</td>
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</tr>
<tr>
<td>• Clear and accurate understanding of how to write a quadratic function in standard form given three points on its graph</td>
<td>• Largely correct understanding of how to write a quadratic function in standard form given three points on its graph</td>
<td>• Partial understanding of how to write a quadratic function in standard form given three points on its graph</td>
<td>• Little or no understanding of how to write a quadratic function in standard form given three points on its graph</td>
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<tr>
<td>• Clear and accurate understanding of how to transform a quadratic function from standard to vertex form</td>
<td>• Largely correct understanding of how to transform a quadratic function from standard to vertex form</td>
<td>• Difficulty with transforming a quadratic function from standard to vertex form</td>
<td>• Little or no understanding of how to transform a quadratic function from standard to vertex form</td>
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<tr>
<td>• Clear and accurate understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola</td>
<td>• Largely correct understanding of how to identify key features of a graph of a parabola and how they relate to the equation for a parabola</td>
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### Problem Solving

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<tr>
<td>• An appropriate and efficient strategy that results in a correct answer</td>
<td>• A strategy that may include unnecessary steps but results in a correct answer</td>
<td>• A strategy that results in some incorrect answers</td>
<td>• No clear strategy when solving problems</td>
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### Mathematical Modeling / Representations

<table>
<thead>
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<th>Mathematical Modeling / Representations (Items 1, 2b, 3b, 4b, 4d)</th>
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<tr>
<td>• Effective understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features</td>
<td>• Adequate understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features</td>
<td>• Partial understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features</td>
<td>• Little or no understanding of how to model real-world scenarios with quadratic functions and parabolas and interpret their key features</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of how to graph quadratic functions using transformations, and how to graph parabolas</td>
<td>• Largely correct understanding of how to graph quadratic functions using transformations, and how to graph parabolas</td>
<td>• Some difficulty with understanding how to graph quadratic functions using transformations and with graphing parabolas</td>
<td>• Inaccurate or incomplete understanding of how to graph quadratic functions using transformations, and how to graph parabolas</td>
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### Reasoning and Communication

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<tr>
<td>• Precise use of appropriate math terms and language to describe how to graph a quadratic function as a transformation of ( f(x) = x^2 )</td>
<td>• Adequate explanation of how features of a graph relate to a real-world scenario</td>
<td>• Misleading or confusing descriptions of how to graph a quadratic function as a transformation of ( f(x) = x^2 )</td>
<td>• Incomplete or inaccurate explanations of how to graph a quadratic function as a transformation of ( f(x) = x^2 )</td>
<td></td>
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<tr>
<td>• Precise use of appropriate math terms and language to explain how features of a graph relate to a real-world scenario</td>
<td>• Adequate descriptions of how to graph a quadratic function as a transformation of ( f(x) = x^2 )</td>
<td>• Partially correct explanation of how features of a graph relate to a real-world scenario</td>
<td>• Incorrect or incomplete explanation of how features of a graph relate to a real-world scenario</td>
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</table>
Ms. Picasso, sponsor for her school’s art club, sells calendars featuring student artwork to raise money for art supplies. A local print shop sponsors the calendar sale and donates the printing and supplies. From past experience, Ms. Picasso knows that she can sell 150 calendars for $3.00 each. She considers raising the price to try to increase the profit that the club can earn from the sale. However, she realizes that by raising the price, the club will sell fewer than 150 calendars.

1. If Ms. Picasso raises the price of the calendar by x dollars, write an expression for the price of one calendar.
   \[ P = 3 + x \]

2. In previous years, Ms. Picasso found that for each $0.40 increase in price, the number of calendars sold decreased by 10. Complete the table below to show that relationship between the price increase and the number of calendars sold.

<table>
<thead>
<tr>
<th>Increase in price ((x), $)</th>
<th>Number of calendars sold</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00</td>
<td>150</td>
</tr>
<tr>
<td>0.40</td>
<td>140</td>
</tr>
<tr>
<td>0.80</td>
<td>130</td>
</tr>
<tr>
<td>1.20</td>
<td>120</td>
</tr>
</tbody>
</table>

3. Model with mathematics. Use the data in the table to write an expression that models the number of calendars sold in terms of \(x\), the price increase.
   \[ 150 - 25x \]

4. Write a function that models \(A(x)\), the amount of money raised selling calendars when the price is increased \(x\) dollars.
   \[ A(x) = (3 + x)(150 - 25x) \]

**ACTIVITY 12**

**Common Core State Standards for Activity 12**

- HSF-IF.B.4 For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.
- HSF-IF.B.5 Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.
6. Write your function $A(x)$ in standard form. Identify the constants $a$, $b$, and $c$.

$$A(x) = -25x^2 + 75x + 450; \quad a = -25, \quad b = 75, \quad c = 450$$

6. Graph $A(x)$ on the coordinate grid.

7. a. For what values of $x$ does the value of $A(x)$ increase as you move from left to right on the graph?
   The value of $A(x)$ increases for values of $x$ less than 1.5.

   b. For what values of $x$ does the value of $A(x)$ decrease as you move from left to right on the graph?
   The value of $A(x)$ decreases for values of $x$ greater than 1.5.

8. Reason quantitatively. Based on the model, what is the maximum amount of money that can be earned? What is the increase in price of a calendar that will yield that maximum amount of money?

   Maximum amount of money that can be earned is $506.25, and the increase in price that yields this maximum is $1.50.
9. a. What feature of the graph gives the information that you used to answer Item 8?
   Answers may vary but must identify the “highest point” on the graph of the function, the vertex.

   b. How does this feature relate to the intervals of $x$ for which $A(x)$ is increasing and decreasing?
   The vertex (or highest point) separates the two intervals. From left to right on the graph, the value of $A(x)$ increases until it reaches the vertex and then decreases.

The point that represents the maximum value of $A(x)$ is the vertex of this parabola. The $x$-coordinate of the vertex of the graph of $f(x) = ax^2 + bx + c$ can be found using the formula $x = -\frac{b}{2a}$.

10. Use this formula to find the $x$-coordinate of the vertex of $A(x)$.

   $-\frac{75}{2(-25)} = -\frac{75}{-50} = 1.5$

**Check Your Understanding**

11. Look back at the expression you wrote for $A(x)$ in Item 4. Explain what each part of the expression equal to $A(x)$ represents.

12. Is the vertex of the graph of a quadratic function always the highest point? Explain.

13. The graph of a quadratic function $f(x)$ opens upward, and its vertex is $(-2, 5)$. For what values of $x$ is the value of $f(x)$ increasing? For what values of $x$ is the value of $f(x)$ decreasing? Explain your answers.

14. **Construct viable arguments.** Suppose you are asked to find the vertex of the graph of $f(x) = -3(x - 4)^2 + 1$. Which method would you use? Explain why you would choose that method.
LEsson 12-1 PRACTICE

Mr. Picasso would like to create a small rectangular vegetable garden adjacent to his house. He has 24 ft of fencing to put around three sides of the garden.

15. Construct viable arguments. Explain why $24 - 2x$ is an appropriate expression for the length of the garden in feet given that the width of the garden is $x$ ft.

16. Write the standard form of a quadratic function $G(x)$ that gives the area of the garden in square feet in terms of $x$. Then graph $G(x)$.

17. What is the vertex of the graph of $G(x)$? What do the coordinates of the vertex represent in this situation?

18. Reason quantitatively. What are the dimensions of the garden that yield that maximum area? Explain your answer.

Write each quadratic function in standard form and identify the vertex.

19. $f(x) = (3x - 6)(x + 4)$

20. $f(x) = 2(x - 6)(20 - 3x)$
Lesson 12-2
More Key Features of Quadratic Functions

Learning Targets:
- Write a quadratic function from a verbal description.
- Identify and interpret key features of the graph of a quadratic function.

SUGGESTED LEARNING STRATEGIES: Interactive Word Wall, Quickwrite, Think Aloud, Discussion Groups, Self Revision/Peer Revision

An intercept occurs at the point of intersection of a graph and one of the axes. For a function \( f \), an \( x \)-intercept is a value \( n \) for which \( f(n) = 0 \). The \( y \)-intercept is the value of \( f(0) \). Use the graph that you made in Item 6 in the previous lesson for Items 1 and 2 below.

1. What is the \( y \)-intercept of the graph of \( A(x) \)? What is the significance of the \( y \)-intercept in terms of the calendar problem?
   The \( y \)-intercept of the graph is 450. This represents the amount earned if the price is increased $0.

2. Make sense of problems. What are the \( x \)-intercepts of the graph of \( A(x) \)? What is the significance of each \( x \)-intercept in terms of the calendar problem?
   The \( x \)-intercepts are −3 and 6. −3 represents a decrease in price of $3 that will yield no profit (the calendars are free). 6 represents an increase in $6 in price that will yield no profit, meaning no calendars sold.

3. The \( x \)-intercepts of the graph of \( f(x) = ax^2 + bx + c \) can be found by solving the equation \( ax^2 + bx + c = 0 \). Solve the equation \( A(x) = 0 \) to verify the \( x \)-intercepts of the graph.
   \(-25x^2 + 75x + 450 = 0 \) factors to \(-25(x + 3)(x - 6) = 0 \). The solutions are \( x = -3 \), \( x = 6 \).

4. a. Recall that \( x \) represents the increase in the price of the calendars.
   Explain what negative values of \( x \) represent in this situation.
   A negative value of \( x \) would indicate a decrease in the price of the calendars. For example, an \( x \)-value of −1 represents a $1 decrease in the price of the calendars.

   b. Recall that \( A(x) \) represents the amount of money raised from selling the calendars. Explain what negative values of \( A(x) \) represent in this situation.
   A negative value of \( A(x) \) would represent a loss of money from the calendar sales. For example, a value of −1 for \( A(x) \) would indicate that the club lost $1 by selling the calendars. However, this value occurs only when the price of the calendar is reduced below 0, which does not make sense.

4–6 Chunking the Activity, Activating Prior Knowledge, Group Presentation
For Items 4–6, place students in small groups of varying abilities to provide them with opportunities to further explore the interpretations of the graphs together.

Item 4 addresses what it means to have a negative \( x \)-value and a negative \( A(x) \)-value. For part b, ask students to determine whether there is a price increase for the calendars that would also result in a negative value of \( A(x) \). They should find that an increase of more than $6 would cause this.

ACTIVITY 12 Continued

Lesson 12-2

PLAN

Pacing: 1 class period

Chunking the Lesson
#1–2 #3
#4–6 #7
Check Your Understanding
Lesson Practice

TEACH

Bell-Ringer Activity
Have students find the \( x \)-intercepts and \( y \)-intercepts of the following linear equations. This will help prepare them for finding intercepts of quadratic equations.

1. \( x + 2y = 8 \)
   \([x \text{-intercept} = 8; y \text{-intercept} = 4]\)

2. \( 2x - 6y = 12 \)
   \([x \text{-intercept} = 6; y \text{-intercept} = -2]\)

3. \( -5x + y = 10 \)
   \([x \text{-intercept} = -2; y \text{-intercept} = 10]\)

Developing Math Language
When written as ordered pairs, \( x \)-intercepts represent the value of the equation when \( y = 0 \). Therefore, \( x \)-intercepts are always of the form \((x, 0)\) when written as an ordered pair. When written as ordered pairs, \( y \)-intercepts represent the value of the equation when \( x = 0 \). Therefore, \( y \)-intercepts are always of the form \((0, y)\), when written as an ordered pair.

Additionally, \( x \)-intercepts and \( y \)-intercepts can both be written in function notation. In function notation, the \( x \)-intercept for a function \( f \) is a value for which \( f(x) = 0 \).

The \( y \)-intercept is the value of \( f(0) \).

1–2 Create Representations, Quickwrite
While students may find the intercepts easily, they may have difficulty with the interpretations in the problem context. Be sure that students recognize that the negative \( x \)-intercept actually represents a decrease in price.

3 Create Representations
The intent of Item 3 is that students find the \( x \)-intercepts algebraically to verify the answer found in Item 2.
4–6 (continued) Item 5 asks students to find a reasonable domain for the function, where the graph is above a zero profit. Item 6 asks students to find a reasonable range with the assumption that a profit is made. Both Items 5 and 6 ask students to use prior knowledge by providing answers in inequality notation, interval notation, and set notation. After they have had some time to collaborate, have students present and explain their solutions to the class.

Note that the wording in Item 6a is “assuming that the club makes a profit” which excludes a profit of 0. Later, in Activity Practice Item 14, the wording is “assuming that the club does not want to lose money” which does not exclude a profit of 0.

ELL Support

Explain to students that the use of the word reasonable, when referring to the domain and range in Items 5 and 6, means sensible. In other words, it does not make sense to include a number of calendars in the domain, where the profit is ≤ 0. That is why it is between the x-intercepts (noninclusive). It does not make sense to include anything other than a positive range to represent profit; therefore, the range spans from anything greater than zero up to the maximum point on the parabola, 506.25.

7 Activating Prior Knowledge, Quickwrite, Debriefing

Prior knowledge of line symmetry is necessary to answer this item. Students should recognize that the x-intercepts are equidistant from the axis of symmetry.

Check Your Understanding

Debrief students’ answers to these items to ensure that they understand concepts related to intercepts of quadratic functions.

Answers

8. Sample answer: The amount the club will make is the product of the cost per calendar and the number of calendars sold. Both of these factors depend on x, the increase in price of the calendars. The product of 2 factors that contain x will have an x²-term. So, the amount the club will make is a quadratic function in terms of x.

5. a. Reason quantitatively. What is a reasonable domain of \( A(x) \), assuming that the club makes a profit from the calendar sales? Write the domain as an inequality, in interval notation, and in set notation. 
\[-3 < x < 6; (-3, 6); \{x \mid x \in \mathbb{R}, -3 < x < 6\}\]

b. Explain how you determined the reasonable domain.

A positive value of \( A(x) \) represents a profit for the club, so the reasonable domain includes only the values of \( x \) for which \( A(x) \) is positive. \( A(x) \) is positive only when \( x \) is greater than \(-3\) and less than \( 6\).

6. a. What is a reasonable range of \( A(x) \), assuming that the club makes a profit from the calendar sales? Write the range as an inequality, in interval notation, and in set notation. 
\[
0 < y \leq 506.25; (0, 506.25]; \{y \mid y \in \mathbb{R}, 0 < y \leq 506.25\}
\]

b. Explain how you determined the reasonable range.

A positive value of \( A(x) \) represents a profit for the club, so the value of \( A(x) \) must be greater than 0. The graph and the function rule show that the maximum value of \( A(x) \) is 506.25. Thus, the reasonable range includes values greater than 0 and no more than 506.25.

7. What is the average of the x-intercepts in Item 2? How does this relate to the symmetry of a parabola?

The average of 6 and \(-3\) is 1.5, the x-coordinate of the vertex. The axis of symmetry is a vertical line through the vertex. Therefore, a point on one side of the axis of symmetry will have a corresponding point on the other side the same distance away. This is true of the x-intercepts.
Lesson 12-2
More Key Features of Quadratic Functions

Check Your Understanding

8. Construct viable arguments. Explain why a quadratic function is an appropriate model for the amount the club will make from selling calendars.

9. Can a function have more than one y-intercept? Explain.

10. Do all quadratic functions have two x-intercepts? Explain.

11. Reason abstractly. Explain how the reasonable domain of a quadratic function helps to determine its reasonable range.

LESSON 12-2 PRACTICE

Ms. Picasso is also considering having the students in the art club make and sell candles to raise money for supplies. The function \( P(x) = -20x^2 + 320x - 780 \) models the profit the club would make by selling the candles for \( x \) dollars each.

12. What is the y-intercept of the graph of \( P(x) \), and what is its significance in this situation?

13. What are the x-intercepts of the graph of \( P(x) \), and what is their significance in this situation?

14. Give the reasonable domain and range of \( P(x) \), assuming that the club does not want to lose money by selling the candles. Explain how you determined the reasonable domain and range.

15. Make sense of problems. What selling price for the candles would maximize the club’s profit? Explain your answer.

Identify the x- and y-intercepts of each function.

16. \( f(x) = x^2 + 11x + 30 \)  

17. \( f(x) = 4x^2 + 14x - 8 \)

CONNECT TO TECHNOLOGY

When answering Items 12–15, it may help you to view a graph of the function on a graphing calculator.

Answers

9. No. Sample explanation: If a graph of a relationship has more than one y-intercept, then the vertical line \( x = 0 \) would intersect the graph at more than one point. If the graph of a relationship fails the vertical line test, then it is not a function.

10. No. If the vertex of the graph of a quadratic function is on the x-axis, then the function has only one x-intercept. If the graph of a quadratic function opens upward and its vertex is above the x-axis, then the function has no x-intercepts. Similarly, if the graph of a quadratic function opens downward and its vertex is below the x-axis, then the function has no x-intercepts.

11. The reasonable domain includes only the values of \( x \) that make sense as inputs for the quadratic function in the given situation. The reasonable domain restricts the range to the values of the function for those values of \( x \). (Note that there may be restrictions on the reasonable range other than those having to do with the domain.)
**Bell-Ringer Activity**

In order to help students with Example A, review the following key elements that describe graphs of quadratic functions in the form \( f(x) = ax^2 + bx + c \).

- If \( a > 0 \), the graph opens upward.
- If \( a < 0 \), the graph opens downward.
- If \( |a| > 1 \), it will be narrower than the parent function of \( y = x^2 \).
- If \( |a| < 1 \), it will be wider than the parent function of \( y = x^2 \).
- The axis of symmetry is \( x = -\frac{b}{2a} \).
- The vertex has an \( x \)-coordinate of \( -\frac{b}{2a} \).
- The \( y \)-intercept is \( c \). Therefore, the point \( (0, c) \) is on the parabola.

**Example A Create Representations, Group Presentation, Debriefing**

As shown by the items in Try These A, functions may have irrational \( x \)-intercepts or no \( x \)-intercepts. This may initiate discussion that will enable students to make some connections to prior learning regarding discriminants of a quadratic equation. Thorough debriefing and group presentations should follow this Example and Try These items. Students should find that the graph in Try These Item d has no \( x \)-intercepts because \( f(x) = 0 \) has no real solutions. Because of this, students may not immediately see how to draw the parabola, because they have only two points. Use your questioning skills to help them realize that another point can be found by reflecting the point containing the \( y \)-intercept over the axis of symmetry.

**MATH TIP**

The graph of the function \( f(x) = ax^2 + bx + c \) will open upward if \( a > 0 \) and open downward if \( a < 0 \).

If the parabola opens up, then the \( y \)-coordinate of the vertex is the minimum value of the function. If it opens down, the \( y \)-coordinate of the vertex is the maximum value of the function.

**Example A**

For the quadratic function \( f(x) = 2x^2 - 9x + 4 \), identify the vertex, the \( y \)-intercept, \( x \)-intercept(s), and the axis of symmetry. Graph the function.

Identify \( a, b, \) and \( c \).  

\[ a = 2, \ b = -9, \ c = 4 \]

**Vertex**

Use \( -\frac{b}{2a} \) to find the \( x \)-coordinate of the vertex.

\[ \frac{-(-9)}{2(2)} = \frac{9}{4} \]

Then use \( f\left(\frac{-b}{2a}\right) \) to find the \( y \)-coordinate.

\[ f\left(\frac{9}{4}\right) = \left(\frac{9}{4}\right)^2 - 9\left(\frac{9}{4}\right) + 4 \]

**\( y \)-intercept**

Evaluate \( f(x) \) at \( x = 0 \).

\[ f(0) = 4 \]

**\( x \)-intercepts**

Let \( f(x) = 0 \).

\[ 2x^2 - 9x + 4 = 0 \]

Then solve for \( x \) by factoring or by using the Quadratic Formula.

\[ x = \frac{1}{2} \] and \( x = 4 \) are solutions, so \( x \)-intercepts are \( \frac{1}{2} \) and 4.

**Axis of Symmetry**

Find the vertical line through the vertex, \( x = -\frac{b}{2a} \).

\[ x = \frac{9}{4} \]

**Graph**

Graph the points identified above: vertex, point on \( y \)-axis, points on \( x \)-axis.

Then draw the smooth curve of a parabola through the points.

The \( y \)-coordinate of the vertex represents the minimum value of the function. The minimum value is \(-\frac{49}{8}\).
Lesson 12-3
Graphing Quadratic Functions

Try These A
For each quadratic function, identify the vertex, the y-intercept, the x-intercept(s), and the axis of symmetry. Then graph the function and classify the vertex as a maximum or minimum.

a. \( f(x) = x^2 - 4x - 5 \)
   - vertex: (2, -9)
   - y-intercept: -5
   - x-intercepts: -1, 5
   - axis of symmetry: \( x = 2 \)
   - vertex is a minimum

b. \( f(x) = -3x^2 + 8x + 16 \)
   - vertex: \( \left(\frac{4}{3}, \frac{64}{3}\right) \)
   - y-intercept: 16
   - x-intercepts: -4/3, 4
   - axis of symmetry: \( x = \frac{4}{3} \)
   - vertex is a maximum

c. \( f(x) = 2x^2 + 8x + 3 \)
   - vertex: (-2, -5)
   - y-intercept: 3
   - x-intercepts: \(-2 - \frac{1}{2}\sqrt{10}, -2 + \frac{1}{2}\sqrt{10}\) (rounded)
   - axis of symmetry: \( x = -2 \)
   - vertex is a minimum

d. \( f(x) = -x^2 + 4x - 7 \)
   - vertex: (2, -3)
   - y-intercept: -7
   - x-intercepts: none
   - axis of symmetry: \( x = 2 \)
   - vertex is a maximum

Consider the calendar fund-raising function from Lesson 12-1, Item 5, \( A(x) = -25x^2 + 75x + 450 \), whose graph is below.

1. Make sense of problems. Suppose that Ms. Picasso raises $450 in the calendar sale. By how much did she increase the price? Explain your answer graphically and algebraically.

   Price increase is either $0 (represented by the y-intercept) or $3. Algebraically, the solutions are found by solving the equation \( 450 = -25x^2 + 75x + 450 \). Graphically, they are the x-coordinates of the two points on the graph that have a y-coordinate of 450.

   - a.
   - b.
   - c. \( f(x) = 2x^2 + 8x + 3 \)
   - d. \( f(x) = -x^2 + 4x - 7 \)

MATH TIP

Quadratic equations may be solved by algebraic methods such as factoring or the Quadratic Formula. An equation can be solved on a graphing calculator by entering each side of the equation as a function, graphing both functions, and finding the points of intersection. The x-coordinates of the intersection points are the solutions.

Differentiating Instruction

Some students may confuse the maximum or minimum value of the quadratic function with the x-coordinate of the vertex. Emphasize that the maximum or minimum value is actually the y-coordinate that corresponds with the x-coordinate of \( -\frac{b}{2a} \).

1–4 Identify a Subtask, Create Representations, Quickwrite, Debriefing

At first, for Item 1, students may think that the $450 amount is only possible with a $0 increase. Guide students toward a graphic solution to point out that there are two possible solutions.
2. Suppose Ms. Picasso wants to raise $600. Describe why this is not possible, both graphically and algebraically. **Raising $600 is not possible because solutions to the equation $600 = −25x^2 + 75x + 450$ are complex. Graphically, no point on the graph has a $y$-coordinate of 600.**

3. In Lesson 12-1, Item 8, you found that the maximum amount of money that could be raised was $506.25. Explain both graphically and algebraically why this is true for only one possible price increase. **Solving the equation $506.25 = −25x^2 + 75x + 450$ yields $(2x - 3)^2 = 0$ with only one solution, $x = 1.5$. Graphically, there is only one point on the graph with this $x$-value.**

4. **Reason quantitatively.** What price increase would yield $500 in the calendar sale? Explain how you determined your solution. **The price increase that will yield $500 in the calendar sale is either $1 or $2. This can be solved either algebraically or graphically.**
**Lesson 12-3**  
**Graphing Quadratic Functions**

**Check Your Understanding**

5. **Make use of structure.** If you are given the equation of a quadratic function in standard form, how can you determine whether the function has a minimum or maximum?

6. Explain how to find the x-intercepts of the quadratic function \( f(x) = x^2 + 17x + 72 \) without graphing the function.

7. Explain the relationships among these features of the graph of a quadratic function: the vertex, the axis of symmetry, and the minimum or maximum value.

**LESSON 12-3 PRACTICE**

Recall that the function \( P(x) = -20x^2 + 320x - 780 \) models the profit the art club would make by selling candles for \( x \) dollars each. The graph of the function is below.

![Graph of Profit Model for Selling Candles](image)

8. Based on the model, what selling price(s) would result in a profit of $320? Explain how you determined your answer.

9. **Construct viable arguments.** Could the club make $600 in profit by selling candles? Justify your answer both graphically and algebraically.

10. If the club sells the candles for $6 each, how much profit can it expect to make? Explain how you determined your answer.

For each function, identify the vertex, \( y \)-intercept, x-intercept(s), and axis of symmetry. Graph the function. Identify whether the function has a maximum or minimum and give its value.

11. \( f(x) = -x^2 + x + 12 \)

12. \( g(x) = 2x^2 - 11x + 15 \)

**LESSON 12-3 PRACTICE**

8. $5 and $11; Sample explanation: Set \( P(x) \) equal to 320: \( 320 = -20x^2 + 320x - 780 \). Subtract 320 from both sides to get \( 0 = -20x^2 + 320x - 1100 \). Factor: \( 0 = -20(x - 10)(x - 11) \). So, \( x = 10 \) or \( x = 11 \), which means that a selling price of $5 or $11 will result in a profit of $320.

9. No. Graphically: The graph shows that the vertex of the profit function is (8, 500), so the maximum profit the club can earn is $500. Algebraically: Set \( P(x) \) equal to 600: \( 600 = -20x^2 + 320x - 780 \). Subtract 600 from both sides to write the equation in standard form: \( 0 = -20x^2 + 300x - 1380 \). Use the Quadratic Formula to solve for \( x \), which shows that \( x = 8 \pm \sqrt{5} \). Because the equation has complex solutions, there is no real value of \( x \) that results in a profit of $600.

10. $420. Sample explanation: Evaluate \( P(x) \) for \( x = 6 \): \( P(6) = -20(6)^2 + 320(6) - 780 = 420 \). The club can expect to make a profit of $420 if it sells the candles for $6 each.

11. Vertex is \( (1, 12) \); \( y \)-intercept is 12; x-intercepts are -3 and 4; axis of symmetry is \( x = \frac{1}{2} \); maximum value is \( 12 \). Check students’ graphs.

12. Vertex is \( (2, 1) \); \( y \)-intercept is 15; x-intercepts are 1 and 3; axis of symmetry is \( x = 2 \); minimum value is \( 1 \). Check students’ graphs.

**ACTIVITY 12 Continued**

**Check Your Understanding**

Debrief students’ answers to these items to ensure that they understand concepts related to key features of quadratic functions.

**Answers**

5. Look at the coefficient of the \( x^2 \)-term. If the coefficient is positive, the graph of the function opens upward, and the function has a minimum. If the coefficient is negative, the graph of the function opens downward, and the function has a maximum.

6. Set \( f(x) = 0 \). Then solve the resulting equation, \( 0 = x^2 + 17x + 72 \), for \( x \). The right side of the equation can be factored: \( 0 = (x + 9)(x + 8) \), and the solutions are \( x = -9 \) and \( x = -8 \), which means that the x-intercepts of the function are -9 and -8.

7. The vertex of a quadratic function \( f(x) = ax^2 + bx + c \) is given by \( \left( -\frac{b}{2a}, f\left( -\frac{b}{2a} \right) \right) \). The axis of symmetry is the vertical line used to determine the equation of the axis of symmetry. Thus, the axis of symmetry is the line \( x = -\frac{b}{2a} \).

The minimum or maximum value is the value of the function at the vertex, given by the \( y \)-coordinate of the vertex. Thus, the minimum or maximum value is \( f\left( -\frac{b}{2a} \right) \).

**ASSESS**

Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

**ADAPT**

Check students’ answers to the Lesson Practice to ensure that they understand how to graph a quadratic function. Remind students to first find the vertex and intercepts, and determine if the vertex is a maximum or a minimum. Students should also be able to use the graph and answer questions about the function. For students who need extra help, pair them with a partner who is proficient at the task to complete additional practice problems.
ACTIVITY 12  continued

Learning Targets:
- Use the discriminant to determine the nature of the solutions of a quadratic equation.
- Use the discriminant to help graph a quadratic function.

SUGGESTED LEARNING STRATEGIES: Summarizing, Note Taking, Create Representations, Quickwrite, Self Revision/Peer Revision

The discriminant of a quadratic equation \( ax^2 + bx + c = 0 \) can determine not only the nature of the solutions of the equation, but also the number of \( x \)-intercepts of its related function \( f(x) = ax^2 + bx + c \).

<table>
<thead>
<tr>
<th>Discriminant of ( ax^2 + bx + c = 0 )</th>
<th>Solutions and ( x )-intercepts</th>
<th>Sample Graph of ( f(x) = ax^2 + bx + c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( b^2 - 4ac &gt; 0 )</td>
<td>Two real solutions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Two ( x )-intercepts</td>
<td></td>
</tr>
<tr>
<td></td>
<td>roots are rational</td>
<td></td>
</tr>
<tr>
<td></td>
<td>roots are irrational</td>
<td></td>
</tr>
<tr>
<td>( b^2 - 4ac = 0 )</td>
<td>One real, rational solution (a double root)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>One ( x )-intercept</td>
<td></td>
</tr>
<tr>
<td>( b^2 - 4ac &lt; 0 )</td>
<td>Two complex conjugate solutions</td>
<td></td>
</tr>
<tr>
<td></td>
<td>No ( x )-intercepts</td>
<td></td>
</tr>
</tbody>
</table>

Discriminant is 81; 2 real, rational roots; \( x \)-intercepts are \(-2.5 \) and 2.

Discriminant is 5; 2 real, irrational roots; \( x \)-intercepts are \(-\frac{3}{2} \pm \frac{\sqrt{5}}{2} \) (approximately \(-2.62 \) and \(-0.38 \)).

Answers
1. Discriminant is 0; 1 real (double) root; \( x \)-intercept is \(-1.5 \).
2. Discriminant is \(-39 \); no real roots (complex conjugates); no \( x \)-intercepts.
3. Discriminant is \( 39 \); 2 real, rational roots; \( x \)-intercepts are \(-2.5 \) and 2.
4. Discriminant is 5; 2 real, irrational roots; \( x \)-intercepts are \(-\frac{3}{2} \pm \frac{\sqrt{5}}{2} \) (approximately \(-2.62 \) and \(-0.38 \)).
Lesson 12-4
The Discriminant

Check Your Understanding

For each equation, find the value of the discriminant and describe the nature of the solutions. Then graph the related function and find the x-intercepts.

1. $4x^2 + 12x + 9 = 0$
2. $2x^2 + x + 5 = 0$
3. $2x^2 + x - 10 = 0$
4. $x^2 + 3x + 1 = 0$

5. Reason abstractly. How can calculating the discriminant help you decide whether to use factoring to solve a quadratic equation?

6. The graph of a quadratic function $f(x)$ is shown at right. Based on the graph, what can you conclude about the value of the discriminant and the nature of the solutions of the related quadratic equation? Explain.

7. $x^2 - 4x + 1 = 0$
8. $x^2 - 6x + 15 = 0$
9. $4x^2 + 4x + 1 = 0$
10. $x^2 - 2x - 15 = 0$

11. Discriminant is $-24$; 2 complex conjugate roots; no x-intercepts.
12. Discriminant is 0; 1 real, rational root (a double root); x-intercept is $-\frac{1}{2}$.
13. Discriminant is 64; 2 rational roots; x-intercepts are $-3$ and 5.

ASSESS
Students’ answers to Lesson Practice problems will provide you with a formative assessment of their understanding of the lesson concepts and their ability to apply their learning. See the Activity Practice for additional problems for this lesson. You may assign the problems here or use them as a culmination for the activity.

LESSON 12-4 PRACTICE

7. A quadratic equation has two rational solutions. How many x-intercepts does the graph of the related quadratic function have? Explain your answer.

8. Make sense of problems. The graph of a quadratic function has one x-intercept. What can you conclude about the value of the discriminant of the related quadratic equation? Explain your reasoning.

9. A quadratic equation has two irrational roots. What can you conclude about the value of the discriminant of the equation?

For each equation, find the value of the discriminant and describe the nature of the solutions. Then graph the related function and find the x-intercepts.

10. $x^2 - 4x + 1 = 0$
11. $x^2 - 6x + 15 = 0$
12. $4x^2 + 4x + 1 = 0$
13. $x^2 - 2x - 15 = 0$

5. If the discriminant is both positive and a perfect square, then the roots of the equation are rational, so the equation can be solved by factoring.

6. The graph of the function has no x-intercepts, so the solutions of the related quadratic equation are complex conjugates and the discriminant is negative.

ADAPT
Check students’ answers to the Lesson Practice to ensure that they understand how to use the discriminant to determine the nature of the roots of an equation. Students can make a graphic organizer to display the information and refer to the organizer until they are proficient in the concept.
Learning Targets:
• Graph a quadratic inequality in two variables.
• Determine the solutions to a quadratic inequality by graphing.

SUGGESTED LEARNING STRATEGIES: Marking the Text, Create Representations, Guess and Check, Think-Pair-Share, Quickwrite

The solutions to quadratic inequalities of the form \( y > ax^2 + bx + c \) or \( y < ax^2 + bx + c \) can be most easily described using a graph. An important part of solving these inequalities is graphing the related quadratic functions.

Example A
Solve \( y > -x^2 - x + 6 \).

Graph the related quadratic function \( y = -x^2 - x + 6 \).

If the inequality symbol is \( > \) or \( < \), use a dotted curve.

If the symbol is \( \geq \) or \( \leq \), then use a solid curve.

This curve divides the plane into two regions.

Test \((0, 0)\) in \( y > -x^2 - x + 6 \).

\[
0 > -0^2 - 0 + 6
\]

\(0 > 6\) is a false statement.

Choose a point on the plane, but not on the curve, to test.

\( (0, 0) \) is an easy point to use, if possible.

If the statement is true, shade the region that contains the point. If it is false, shade the other region.

The shaded region represents all solutions to the quadratic inequality.
Lesson 12-5
Graphing Quadratic Inequalities

Try These A
Solve each inequality by graphing.

a. \( y \geq x^2 + 4x - 5 \)
   (x-intercepts: -5, 1)

b. \( y > 2x^2 - 5x - 12 \)
   (x-intercepts: -1.5, 4)

c. \( y < -3x^2 + 8x + 3 \)
   (x-intercepts: \(-\frac{1}{3}, 3\))

Check Your Understanding

1. The solutions of which inequality are shown in the graph?
   A. \( y \leq -2x^2 + 8x - 7 \)
   B. \( y \geq -2x^2 + 8x - 7 \)
   C. \( y \leq 2x^2 - 8x - 7 \)
   D. \( y \geq 2x^2 - 8x - 7 \)

2. Reason abstractly. How does graphing a quadratic inequality in two variables differ from graphing the related quadratic function?

3. Graph the quadratic inequality \( y \geq -x^2 - 6x - 13 \). Then state whether each ordered pair is a solution of the inequality.
   a. \((-1, -6)\)  b. \((-4, -8)\)  c. \((-6, -10)\)  d. \((-2, -5)\)

Check Your Understanding

Debrief students' answers to these items to ensure that they understand concepts related to solving quadratic inequalities.

Answers
1. A
2. Sample answer: To graph a quadratic inequality in 2 variables, start by graphing the related quadratic function, but use a dotted line for the parabola if the inequality symbol is \(< \) or \(> \). Otherwise, use a solid curve. You must also shade the region inside the parabola or outside the parabola when graphing a quadratic inequality. To decide which region to shade, use a test point.

   Hint: it may be helpful to get in the habit of always choosing a test point from inside the parabola. In the event the test point is a solution, automatically shade inside the parabola. On the other hand, if the test point is not a solution, automatically shade outside the parabola.

   a. solution
   b. not a solution
   c. solution
   d. solution
Lesson 12-5
Graphing Quadratic Inequalities

LESSON 12-5 PRACTICE

Graph each inequality.

4. \[ y \leq x^2 + 4x + 7 \]
5. \[ y < x^2 - 6x + 10 \]
6. \[ y > \frac{1}{2} x^2 + 2x + 1 \]
7. \[ y \geq -2x^2 + 4x + 1 \]

8. Construct viable arguments. Give the coordinates of two points that are solutions of the inequality \[ y \leq x^2 - 6x + 4 \] and the coordinates of two points that are not solutions of the inequality. Explain how you found your answers.

9. Model with mathematics. The students in Ms. Picasso's art club decide to sell candles in the shape of square prisms. The height of each candle will be no more than 10 cm. Write an inequality to model the possible volumes in cubic centimeters of a candle with a base side length of \( x \) cm.

10. Make sense of problems. Brendan has 400 cm\(^3\) of wax. Can he make a candle with a base side length of 6 cm that will use all of the wax if the height is limited to 10 cm? Explain your answer using your inequality from Item 9.
**ACTIVITY 12 PRACTICE**

Write your answers on notebook paper. Show your work.

**Lesson 12-1**

The cost of tickets to a whale-watching tour depends on the number of people in the group. For each additional person, the cost per ticket decreases by $1. For a group with only two people, the cost per ticket is $44. Use this information for Items 1–7.

1. Complete the table below to show the relationship between the number of people in a group and the cost per ticket.

<table>
<thead>
<tr>
<th>Number of People</th>
<th>Cost per Ticket ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>46</td>
</tr>
<tr>
<td>3</td>
<td>43</td>
</tr>
<tr>
<td>4</td>
<td>42</td>
</tr>
<tr>
<td>5</td>
<td>41</td>
</tr>
</tbody>
</table>

2. Use the data in the table to write an expression that models the cost per ticket in terms of the number of people in a group.

3. Write a quadratic function in standard form that models \( T(x) \), the total cost of the tickets for a group with \( x \) people.

4. Graph \( T(x) \) on a coordinate grid.

5. a. For what values of \( x \) does the value of \( T(x) \) increase as you move from left to right on the graph?

   b. For what values of \( x \) does the value of \( T(x) \) decrease as you move from left to right on the graph?

6. What is the vertex of the graph of \( T(x) \)? What are the coordinates of the vertex in this situation?

7. Groups on the tour are limited to a maximum size of 20 people. What is the total cost of the tickets for a group of 20 people? Explain how you found your answer.

Write each quadratic function in standard form and identify the vertex.

8. \( f(x) = 4x^2 - 4(x + 5) \)

9. \( f(x) = 4(x + 8)(10 - x) \)

**Lesson 12-2**

Mr. Gonzales would like to create a playground in his backyard. He has 20 ft of fencing to enclose the play area. Use this information for Items 10–13.

10. Write a quadratic function in standard form that models \( f(x) \), the total area of the playground in square feet in terms of its width \( x \) ft. Then graph \( f(x) \).

11. Write the \( x \)- and \( y \)-intercepts of \( f(x) \) and interpret them in terms of the problem.

12. Give the reasonable domain and range of \( f(x) \) as inequalities, in interval notation, and in set notation. Explain how you determined the reasonable domain and range.

13. What is the maximum area for the playground? What are the dimensions of the playground with the maximum area? Identify the \( x \)- and \( y \)-intercepts of each function.

14. \( f(x) = x^2 + 3x - 28 \)

15. \( f(x) = 2x^2 + 13x + 15 \)

**Lesson 12-3**

For each function, identify the vertex, \( y \)-intercept, \( x \)-intercept(s), and axis of symmetry. Identify whether the function has a maximum or minimum and give its value.

16. \( f(x) = -x^2 + 4x + 5 \)

17. \( f(x) = 2x^2 - 12x + 13 \)

18. \( f(x) = -3x^2 + 12x - 9 \)

14. \( x \)-intercepts: -7 and 4; \( y \)-intercept: -28

15. \( x \)-intercepts: -5 and \(-\frac{3}{2}\); \( y \)-intercept: 15

16. Vertex is (2, 9); \( y \)-intercept is 5; \( x \)-intercepts are -1 and 5; axis of symmetry is \( x = 2 \); maximum value is 9.

17. Vertex is (3, -5); \( y \)-intercept is 13; \( x \)-intercepts are \( 3 \pm \sqrt{10} \); axis of symmetry is \( x = 3 \); minimum value is -5.

18. Vertex is (2, 3); \( y \)-intercept is -9; \( x \)-intercepts are 1 and 3; axis of symmetry is \( x = 2 \); maximum value is 3.

19. Sample explanation: At \( x = 0 \):

\[
 f(0) = 0^2 - 3(0) - 18 = -18.
\]

The \( y \)-intercept is -18.

20. 0.8 s and 2.2 s after the arrow is shot; Sample explanation: Set \( h(t) \) equal to 10: \( 10 = -5t^2 + 15t + 1 \). Subtract 10 from both sides: \( 0 = -5t^2 + 15t - 9 \). Then use the Quadratic Formula to solve for \( t \).

21. Yes. Graphically: The parabola intersects the horizontal line \( y = 12 \) at 2 points in Quadrant I, representing 2 times at which the arrow has a height of 12 m. Algebraically: Set \( h(t) \) equal to 12: \( 12 = -5t^2 + 15t + 1 \). Subtract 12 from both sides: \( 0 = -5t^2 + 15t - 11 \). Then use the Quadratic Formula to solve for \( t \): \( t \approx 1.3 \) or \( t \approx 1.7 \). These are the 2 values of \( t \) for which the height of the arrow will be 12 m.
**ACTIVITY 12** Continued

22. discriminant: 49; 2 rational roots; $x$-intercepts: $-\frac{1}{2}$ and 3

23. discriminant: $-23$; 2 complex conjugate roots; no $x$-intercepts

24. discriminant: 0; 1 real, rational root; $x$-intercept: $-\frac{1}{2}$

25. discriminant: 12; 2 real, irrational roots; $x$-intercepts: $\frac{3}{2} \pm \frac{\sqrt{3}}{2}$

26. D

27. 1 $x$-intercept; The graph of a quadratic function has an $x$-intercept for each real solution, so if a quadratic equation has 1 rational solution, the graph of its related function will have 1 $x$-intercept.

28. The discriminant is negative. If the graph of a quadratic function has no $x$-intercepts, then its related quadratic equation has 2 complex conjugate solutions and its discriminant is negative.

29. 

![Graph](image)

30. 

![Graph](image)

31. 

![Graph](image)

**ADDITIONAL PRACTICE**

If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.

**ACTIVITY 12** continued

19. Explain how to find the $y$-intercept of the quadratic function $f(x) = x^2 - 3x - 18$ without graphing the function.

20. Based on the model, when will the arrow have a height of 10 m? Round times to the nearest tenth of a second. Explain how you determined your answer.

21. Does the arrow reach a height of 12 m? Justify your answer both graphically and algebraically.

22. $2x^2 - 5x - 3 = 0$

23. $3x^2 + x + 2 = 0$

24. $4x^2 + 4x + 1 = 0$

25. $2x^2 + 6x + 3 = 0$

26. A quadratic equation has two distinct rational roots. Which one of the following could be the discriminant of the equation?

   A. $-6$
   B. 0
   C. 20
   D. 64

27. A quadratic equation has one distinct rational solution. How many $x$-intercepts does the graph of the related quadratic function have? Explain your answer.

28. The graph of a quadratic function has no $x$-intercepts. What can you conclude about the value of the discriminant of the related quadratic equation? Explain your reasoning.

**Lesson 12-5**

Graph each quadratic inequality.

29. $y < x^2 + 7x + 10$

30. $y \geq 2x^2 + 4x - 1$

31. $y > x^2 - 6x + 9$

32. $y \leq -x^2 + 3x + 4$

33. Which of the following is a solution of the inequality $y > -x^2 - 8x - 12$?

   A. $(-6, 0)$
   B. $(-4, -2)$
   C. $(-3, 1)$
   D. $(-2, 4)$

The time in minutes a factory needs to make $x$ cell phone parts in a single day is modeled by the inequality $y \leq -0.0005x^2 + x + 20$, for the domain $0 \leq x \leq 1000$. Use this information for Items 34–36.

34. a. Is the ordered pair $(200, 100)$ a solution of the inequality? How do you know?

   b. What does the ordered pair $(200, 100)$ represent in this situation?

35. What is the longest it will take the factory to make 600 cell phone parts? Explain how you determined your answer.

36. Can the factory complete an order for 300 parts in 4 hours? Explain.

37. Give the coordinates of two points that are solutions of the inequality $y \leq x^2 - 3x - 10$ and the coordinates of two points that are not solutions of the inequality. Explain how you found your answers.

**Mathematical Practices**

Look for and Make Use of Structure

38. Describe the relationship between solving a quadratic equation and graphing the related quadratic function.

**Graphing Quadratics and Quadratic Inequalities**

**Calendar Art**

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The owner of Salon Ultra Blue is working with a pricing consultant to determine the best price to charge for a basic haircut. The consultant knows that, in general, as the price of a haircut at a salon goes down, demand for haircuts at the salon goes up. In other words, if Salon Ultra Blue decreases its prices, more customers will want to get their hair cut there. Based on the consultant’s research, customers will demand 250 haircuts per week if the price per haircut is $20. For each $5 increase in price, the demand will decrease by 25 haircuts per week.

1. Let the function \( f(x) \) model the quantity of haircuts demanded by customers when the price of haircuts is \( x \) dollars.
   a. Reason quantitatively. What type of function is \( f(x) \)? How do you know?
   The function is linear. Sample explanation: The function has a constant rate of change. For each increase of 5 in the value of \( x \), the value of \( f(x) \) decreases by 25.

   b. Write the equation of \( f(x) \).
   \[ f(x) = -5x + 350 \text{ or equivalent} \]

The price of a haircut not only affects demand, but also affects supply. As the price charged for a haircut increases, cutting hair becomes more profitable. More stylists will want to work at the salon, and they will be willing to work longer hours to provide more haircuts.
The consultant gathered the following data on how the price of haircuts affects the number of haircuts the stylists are willing to supply each week.

<table>
<thead>
<tr>
<th>Price per Haircut ($)</th>
<th>Number of Haircuts Available per Week</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>15</td>
</tr>
<tr>
<td>30</td>
<td>55</td>
</tr>
<tr>
<td>40</td>
<td>115</td>
</tr>
<tr>
<td>50</td>
<td>195</td>
</tr>
</tbody>
</table>

2. The relationship shown in the table is quadratic. Write the equation of a quadratic function \( g(x) \) that models the quantity of haircuts the stylists are willing to supply when the price of haircuts is \( x \) dollars.

\[
g(x) = \frac{1}{10}x^2 - x - 5
\]

3. Model with mathematics. Write a system of two equations in two variables for the demand and supply functions. In each equation, let \( y \) represent the quantity of haircuts and \( x \) represent the price in dollars per haircut.

\[
\begin{align*}
y &= -5x + 350 \\
y &= \frac{1}{10}x^2 - x - 5
\end{align*}
\]

4. Graph the system on the coordinate plane.
Lesson 13-1
Solving a System Graphically

5. Explain how you determine the location of the solutions on the graph in Item 4. 
The intersection points of the graphs of the equations represent the solutions of the system.

6. Explain the relationship of the solution to the demand function \( f(x) \) and the supply function \( g(x) \). 
The x-coordinates of the solutions represent prices at which the demand for haircuts, \( f(x) \), is equal to the supply of haircuts, \( g(x) \).

7. Use the graph to approximate the solutions of the system of equations. 
Answers may vary but should be close to \((-80, 750)\) and \((40, 150)\).

Now use a graphing calculator to make better approximations of the solutions of the system as \( Y_1 \) and \( Y_2 \).

8. Use appropriate tools strategically. 
   a. How can you approximate solutions of a system of two equations in two variables by using a table of values on a graphing calculator? 
      Look for values of \( X \) for which \( Y_1 \approx Y_2 \).

   b. Use the table to approximate the solutions of the system. Find the coordinates of the solutions to the nearest integer. 
      \((-83, 764)\) and \((43, 136)\)

**Technology Tip**
You can change the table settings on a graphing calculator by pressing the [2nd] key with TblSet printed above it. The table start setting (TblStart) lets you change the first value of \( X \) displayed in the table. The table step setting (\( \Delta \text{Tbl} \)) lets you adjust the change in \( X \) between rows of the table.

**CONNECT TO AP**
Note that in the Technology Tip there is a reference made to the table step setting on the graphing calculator as (\( \Delta \text{Tbl} \)). It further explains that this function allows you to adjust the change in \( X \) between rows of the table. The Greek letter delta (\( \Delta \)) is frequently used in calculus. It represents the phrase “the change in.” For example, \( \Delta x \) is interpreted as “the change in the value of the variable \( x \).”

4–5 (continued) For Item 5, explain to students that this is the same concept as when they found solutions to a system of linear equations. The only difference is that now one of the graphs is a parabola. Allow time to discuss the results of these items as a class.

6–7 Debriefing The x-axis represents the price of haircuts. The function \( f(x) \) represents the demand of haircuts for a week, based upon the price. The function \( g(x) \) represents the supply of haircuts for a week, based upon the price. Therefore, the x-values of the points where these two graphs intersect represent the two prices at which the demand and supply are equal. Have students round to the nearest 10 when they approximate solutions.

8 Predict and Confirm, Debriefing The students will take what they already know about the graphs of these two functions to find two ordered pairs where the y-values are closest to each other. When using the graphing calculator to view the table of values corresponding to these two functions, students should know that in the TblSet feature, the smaller the value of (\( \Delta \text{Tbl} \)), the more accurate their solutions will be. Since Item 8b is asking for coordinates to the nearest integer, students should set their (\( \Delta \text{Tbl} \)) to a value of 1. Based upon the graph, students should know approximately where they need to scroll within the list of values in order to find the integer solutions.
9. Next, view a graph of the system of equations on the graphing calculator. Adjust the viewing window as needed so that the intersection points of the graphs of the equations are visible. Then use the intersect feature to approximate the solutions of the system of equations. 

$(−82.84903, 764.24513)$ and $(42.849025, 135.75487)$

10. Explain why one of the solutions you found in Item 9 does not make sense in the context of the supply and demand functions for haircuts at the salon. The variable $x$ represents the price in dollars of a haircut, so it does not make sense in this situation for $x$ to be negative. Therefore, the solution with the negative $x$-value should be ignored.

11. Make sense of problems. Interpret the remaining solution in the context of the situation. The $x$-coordinate of the remaining solution shows that when haircuts are priced at about $42.85, the number of haircuts demanded by customers will equal the number of haircuts that the stylists are willing to supply. The $y$-coordinate of the solution shows that this number of haircuts is about 136 per week.

12. Explain why the solution you described in Item 11 is reasonable. Sample answer: When I substitute 42.85 for $x$ in each equation in the system, I get a value of $y$ that is approximately equal to 136. In addition, a price of $42.85 for a haircut seems realistic. It also seems reasonable that a salon could give 136 haircuts in a week.

13. The pricing consultant recommends that Salon Ultra Blue price its haircuts so that the weekly demand is equal to the weekly supply. Based on this recommendation, how much should the salon charge for a basic haircut? $42.85
Lesson 13-1
Solving a System Graphically

14. **Model with mathematics.** Graph each system of one linear equation and one quadratic equation. For each system, list the number of real solutions.

   a. \[\begin{align*}
   y &= x \\
   y &= x^2 - 2
   \end{align*}\]
   2 real solutions

   b. \[\begin{align*}
   y &= 2x - 3 \\
   y &= x^2 - 2
   \end{align*}\]
   1 real solution

   c. \[\begin{align*}
   y &= 3x - 9 \\
   y &= x^2 - 2
   \end{align*}\]
   0 real solutions

15. Make a conjecture about the possible number of real solutions of a system of two equations that includes one linear equation and one quadratic equation.

   **A system of one linear equation and one quadratic equation may have 0, 1, or 2 real solutions.**
ACTIVITY 13 Continued

18. Answers may vary, but should represent a system of one linear equation and one quadratic equation whose graph has two intersection points on the same side of the vertex of the parabola, or one intersection point at the vertex and one intersection point elsewhere on the parabola. Sample answer: The graph of the system \( \begin{align*}
  y &= 2x \\
  y &= x^2
\end{align*} \) has one intersection point at the vertex of the parabola and one intersection point to the right of the vertex.

19. Sample answer: A graph allows you to identify how many real solutions the system has. It also lets you quickly estimate the coordinates of the real solutions. A table will give more exact values for the real solutions.

LESSON 13-1 PRACTICE

20. \( f(x) = -\frac{2}{5}x + 50 \)
21. \( g(x) = \frac{1}{100}x^2 - \frac{3}{10}x - 1 \)
22. \( \begin{align*}
  y &= -\frac{2}{5}x + 50 \\
  y &= \frac{1}{100}x^2 - \frac{3}{10}x - 1
\end{align*} \)
23. Approximations should be close to \((-76.59, 80.64)\) and \((66.59, 23.36)\).
24. \$66.59; Sample explanation: The solution with a negative value for \(x\), the price in dollars, does not make sense in this situation and should be ignored. The \(x\)-coordinate of the remaining solution shows that when formal hairstyles are priced at approximately \$66.59, the number of formal hairstyles demanded by customers will equal the number that the stylists are willing to supply.
25. Sample answer: When I substitute 66.59 for \(x\) into each equation in the system, I get a value of \(y\) that is approximately equal to 23. In addition, a price of \$66.59 for a formal hairstyle seems realistic.
Lesson 13-2
Solving a System Algebraically

Learning Targets:
- Use substitution to solve a system consisting of a linear and nonlinear equation.
- Determine when a system consisting of a linear and nonlinear equation has no solution.

**SUGGESTED LEARNING STRATEGIES:** Summarizing, Identify a Subtask, Think-Pair-Share, Drafting, Self Revision/Peer Revision

In the last lesson, you approximated the solutions to systems of one linear equation and one quadratic equation by using tables and graphs. You can also solve such systems algebraically, just as you did when solving systems of two linear equations.

**Example A**

The following system represents the supply and demand functions for basic haircuts at Salon Ultra Blue, where \( y \) is the quantity of haircuts demanded or supplied when the price of haircuts is \( x \) dollars. Solve this system algebraically to find the price at which the supply of haircuts equals the demand.

\[
\begin{align*}
\left\{ \begin{array}{l}
y = -5x + 350 \\
y = \frac{1}{10}x^2 - x - 5 \\
\end{array} \right.
\]

**Step 1:** Use substitution to solve for \( x \).

\[
y = -5x + 350 \\
-5x + 350 = \frac{1}{10}x^2 - x - 5 \\
0 = \frac{1}{10}x^2 + 4x - 355 \\
0 = x^2 + 40x - 3550 \\
x = \frac{-40 \pm \sqrt{40^2 - 4(1)(-3550)}}{2(1)} \\
x = -20 \pm 5\sqrt{158} \\
x \approx -82.85 \text{ or } x \approx 42.85
\]

**Step 2:** Substitute each value of \( x \) into one of the original equations to find the corresponding value of \( y \).

\[
\begin{align*}
&y = -5x + 350 \\
&y \approx -5(-82.85) + 350 \\
&y \approx 764 \\
&y = -5x + 350 \\
&y \approx -5(42.85) + 350 \\
&y \approx 136
\end{align*}
\]

**MATH TIP**

In this example, the exact values of \( x \) are irrational. Because \( x \) represents a price in dollars, use a calculator to find rational approximations of \( x \) to two decimal places.

**ACTIVITY 13 Continued**

**Lesson 13-2**

**PLAN**

**Pacing:** 1 class period

**Chunking the Lesson**

**Example A #1–3**

**Check Your Understanding**

**Lesson Practice**

**TEACH**

**Bell-Ringer Activity**

Have students solve each of the systems of linear equations by using the substitution method to help prepare them for solving a system of linear and nonlinear equations.

1. \[
\begin{align*}
x + y &= 10 \\
x &= y + 8
\end{align*}
\]

2. \[
\begin{align*}
a - b &= 32 \\
3a - 8b &= 6
\end{align*}
\]

3. \[
\begin{align*}
x + y &= 8 \\
-3x - y &= -8
\end{align*}
\]

**Differentiating Instruction**

Compare and contrast the possible number of real solutions between a system of two linear equations and a system of two equations that includes one linear equation and one quadratic equation.

Two linear equations
- may have one solution, because they intersect at one point;
- may have no solution, because they do not intersect (parallel lines);
- may intersect and have infinitely many solutions (same line).

One linear and one quadratic equation
- may have one real solution, because they intersect at one point where the line touches the parabola;
- may have no solution, because they do not intersect;
- may have two real solutions, because the line intersects the parabola twice.

**Example A Activating Prior Knowledge, Debriefing**

Explain that the benefit of multiplying both sides of the equation by 10 is to eliminate using fractions in the quadratic formula. This can get very messy and cause unnecessary arithmetic errors. Remind students that if there were more than one fraction present, they would multiply both sides of the equation by the least common denominator.
Step 3: Write the solutions as ordered pairs. The solutions are approximately \((-82.85, 764)\) and \((42.85, 136)\). Ignore the first solution because a negative value of \(x\) does not make sense in this situation.

Solution: The price at which the supply of haircuts equals the demand is $42.85. At this price, customers will demand 136 haircuts, and the stylists will supply them.

Try These A
Solve each system algebraically. Check your answers by substituting each solution into one of the original equations. Show your work.

\[
\begin{align*}
a. \quad &y = -2x - 7 \\
&y = -2x^2 + 4x + 1 \\
&(1, -5), (4, -15)
\end{align*}
\]

\[
\begin{align*}
b. \quad &y = x^2 + 6x + 5 \\
&y = 2x + 1 \\
&(-2, -3)
\end{align*}
\]

\[
\begin{align*}
c. \quad &y = \frac{1}{2} (x + 4)^2 + 5 \\
&y = \frac{17}{2} - x \\
&(-1, \frac{19}{2}), (-9, \frac{35}{2})
\end{align*}
\]

\[
\begin{align*}
d. \quad &y = -4x^2 + 5x - 8 \\
&y = -3x - 24 \\
&\left(1 - \sqrt{5}, -27 + 3\sqrt{5}\right), \left(1 + \sqrt{5}, -27 - 3\sqrt{5}\right)
\end{align*}
\]

1. Use substitution to solve the following system of equations. Show your work.

\[
\begin{align*}
&y = 4x + 24 \\
&y = -x^2 + 18x - 29
\end{align*}
\]

\[
\begin{align*}
&(7 - 2i, 52 - 8i), (7 + 2i, 52 + 8i)
\end{align*}
\]
Lesson 13-2
Solving a System Algebraically

2. Describe the solutions of the system of equations from Item 1. The x- and y-coordinates of each of the two solutions are complex numbers. The system of equations has no real solutions.

3. Use appropriate tools strategically. Confirm that the system of equations from Item 1 has no real solutions by graphing the system on a graphing calculator. How does the graph show that the system has no real solutions? It shows that the graphs of the two equations do not intersect. So, there is no real value of x for which the y-values of the two equations are equal.

Check Your Understanding

4. How does solving a system of one linear equation and one quadratic equation by substitution differ from solving a system of two linear equations by substitution?

5. Reason abstractly. What is an advantage of solving a system of one linear equation and one quadratic equation algebraically rather than by graphing or using a table of values?

6. Write a journal entry in which you explain step by step how to solve the following system by using substitution.
   \[
   \begin{align*}
   y &= 2x^2 - 3x + 6 \\
   y &= -2x + 9
   \end{align*}
   \]

7. Could you solve the system in Item 6 by using elimination rather than substitution? Explain.

8. Explain how you could use the discriminant of a quadratic equation to determine how many real solutions the following system has.
   \[
   \begin{align*}
   y &= 4x - 21 \\
   y &= x^2 - 4x - 5
   \end{align*}
   \]

Check Your Understanding

4. Sample answer: When solving a system of two linear equations, you end up with a linear equation after you solve one equation for y and substitute that expression for y into the other equation. When solving a system of one linear equation and one quadratic equation, you end up with a quadratic equation after you solve one equation for y and substitute that expression for y into the other equation.

5. Sample answer: When you solve the system algebraically, you can find the exact values of the coordinates of the solutions. When you solve the system by graphing or using a table of values, you may only be able to approximate the coordinates of the solutions.

6. See below.

7. Yes. Sample explanation: If you subtract the second equation in the system from the first equation, the variable y is eliminated, leaving the equation 0 = 2x^2 - x - 3. You can then solve this equation for x, which will give the x-values of the solutions of the system of equations.

8. Use substitution to find that 4x - 21 = x^2 - 4x - 5. Then write this equation in standard form: 0 = x^2 - 8x + 16. The discriminant of this quadratic equation is \((-8)^2 - 4(1)(16) = 0\). A discriminant of 0 means that the equation 0 = x^2 - 8x + 16 has only one real solution. The system of equations also has only one real solution.

6. Sample answer:
   First use substitution to solve for x.
   \[
   \begin{align*}
   y &= 2x^2 - 3x + 6 \\
   2x^2 - 3x + 6 &= -2x + 9 \\
   2x^2 - x - 3 &= 0 \\
   (2x - 3)(x + 1) &= 0 \\
   x = \frac{3}{2} \text{ or } x = -1
   \end{align*}
   \]
   Substitute each value of x into one of the original equations to find the corresponding value of y.
   \[
   \begin{align*}
   y &= -2x + 9 \\
   y &= -2(\frac{3}{2}) + 9 \\
   y &= -2(-1) + 9 \\
   y &= 6 \\
   y &= 11
   \end{align*}
   \]
   The solutions are \(\left(\frac{3}{2}, 6\right)\) and \((-1, 11)\).
LESSON 13-2 PRACTICE

Find the real solutions of each system algebraically. Show your work.

9. \( \begin{cases} y = -3x - 8 \\ y = x^2 + 4x + 2 \end{cases} \)

10. \( \begin{cases} y = -2x^2 + 16x - 26 \\ y = 72 - 12x \end{cases} \)

11. \( \begin{cases} y = \frac{1}{4}x^2 - 6x + 1 \\ y = \frac{3}{4}x - \frac{23}{2} \end{cases} \)

12. \( \begin{cases} y = (x - 5)^2 - 3 \\ y = -2x - 3 \end{cases} \)

The owner of Salon Ultra Blue is setting the price for hair highlights. The following system represents the demand and supply functions for hair highlights, where \( y \) is the quantity demanded or supplied per week for a given price \( x \) in dollars.

\( \begin{cases} y = -0.8x + 128 \\ y = 0.03x^2 - 1.5x + 18 \end{cases} \)

13. Use substitution to solve the system of equations.

14. **Attend to precision.** How much should the salon charge for hair highlights so that the weekly demand is equal to the weekly supply? Explain how you determined your answer.

15. Explain why your answer to Item 14 is reasonable.
ACTIVITY 13 PRACTICE

Write your answers on notebook paper. Show your work.

Lesson 13-1

Lori was partway up an escalator when her friend Evie realized that she had Lori’s keys. Evie, who was still on the ground floor, tossed the keys up to Lori. The function \( f(x) = -16x^2 + 25x + 5 \) models the height in feet of the keys \( x \) seconds after they were thrown. Use this information for Items 1–5.

1. When the keys are thrown, Lori’s hands are 9 ft above ground level and moving upward at a rate of 0.75 ft/s. Write the equation of a function \( g(x) \) that gives the height of Lori’s hands compared to ground level \( x \) seconds after the keys are thrown.

2. Write the functions \( f(x) \) and \( g(x) \) as a system of two equations in two variables. In each equation, let \( y \) represent height in feet and \( x \) represent time in seconds.

3. Graph the system of equations, and use the graph to approximate the solutions of the system.

4. How long after the keys are thrown will Lori be able to catch them? Assume that Lori can catch the keys when they are at the same height as her hands. Explain how you determined your answer.

5. Explain why your answer to Item 4 is reasonable.

Solve each system by using a graph or table (answers will be approximate).

\[
\begin{align*}
\begin{align*}
9. & \quad \begin{cases} 
y = 3x^2 + 6x + 4 
y = 0.5x + 8 
\end{cases} \\
10. & \quad \begin{cases} 
y = -2x^2 + 8x - 10 
y = -2x + 4 
\end{cases} \\
11. & \quad \begin{cases} 
y = 24 - 4x 
y = x^2 - 12x + 40 
\end{cases} \\
12. & \quad \text{Which ordered pair is a solution of the system of equations graphed below?}
\end{align*}
\]

A. \((-3, 5)\) \hspace{1cm} B. \((-1, 3)\) \hspace{1cm} C. \((2, 0)\) \hspace{1cm} D. \((3, -5)\)

A parallelogram has a height of \( x \) cm. The length of its base is 4 cm greater than its height. A triangle has the same height as the parallelogram. The length of the triangle’s base is 20 cm.

13. Write a system of two equations in two variables that can be used to determine the values of \( x \) for which the parallelogram and the triangle have the same area.

14. Solve the system by using a graph or table.

15. Interpret the solutions of the system in the context of the situation.

ACTIVITY 13 Continued

ACTIVITY PRACTICE

1. \( g(x) = 0.75x + 9 \)

2. \[
\begin{align*}
y &= -16x^2 + 25x + 5 \\
y &= 0.75x + 9 
\end{align*}
\]

3. Solutions are approximately \((0.2, 9.1)\) and \((1.3, 10.0)\).

4. Lori has two chances to catch the keys: about 0.2 s after they are thrown and about 1.3 s after they are thrown. The \( x \)-values of the solutions of the system represent how long after the keys are thrown that they will be at the same height as Lori’s hands.

5. Sample answer: When I substitute 0.2 for \( x \) into each equation in the system, I get a value of \( y \) that is approximately equal to 9.1. When I substitute 1.3 for \( x \) into each equation in the system, I get a value of \( y \) that is approximately equal to 10.0. In addition, it makes sense that Lori will have two chances to catch the keys: once when they are on their way up and once when they are on their way down.

6. \((3, 4), (7, -4)\)

7. approximately \((-7.2, 3.0), (-1.8, 30.0)\)

8. approximately \((0.8, -0.4), (7.2, -25.6)\)

9. 2 real solutions

10. no real solutions

11. 1 real solution

12. B

13. \[
\begin{align*}
y &= (x + 4)x \\
y &= \frac{1}{2}(20)x 
\end{align*}
\]

14. \((0, 0), (6, 60)\)

15. It does not make sense for a parallelogram or a triangle to have a height of 0 cm, so the solution \((0, 0)\) can be ignored. The solution \((6, 60)\) shows that the parallelogram and the triangle have the same area when the height of each is 6 cm. The area of both the parallelogram and the triangle when their height is 6 cm is 60 cm².
ACTIVITY 13 Continued

ACTIVITY PRACTICE
16. (−3, −10), (0, −7)
17. (5, 16)
18. (5, −1), (1, −25)
19. (−8, 3), (1, −1.5)
20. \( y = x^2 - 2x - 4 \)
21. \( y = -4x - 5 \)
22. \( x = \frac{3}{2} \pm \frac{i\sqrt{19}}{2} \)
23. A
24. Sample answer: Use substitution to solve the system. Substitute the expression for \( y \) from the first equation into the second equation:

\[ \begin{align*}
-2x + 4y &= 3x + 5 \\
0 &= x^2 - x + 5
\end{align*} \]

Use the Quadratic Formula to solve for \( x \):

\[ x = \frac{1}{2} \pm \frac{i\sqrt{19}}{2} \]

The values of \( x \) are complex conjugates, so the system of equations has no real solutions.
25. \( f(x) = 0.75x \)
26. \( g(x) = 0.015x^2 + 0.15x \)
27. \( (0, 0), (40, 30) \)
28. If the length of the longest side is 40 in., the charge for nonglare glass will be the same as the charge for regular glass. This charge will be $30. Sample explanation: The \( x \)-coordinates of the solutions of the system represent lengths for which the charges for the two types of glass will be equal. Because the length of a piece of glass must be greater than 0 in., the solution (0, 0) can be ignored. The solution is (40, 30), meaning when the longest side is 40 in., the charge for both types of glass will be $30.
29. \( y = 200 + 8x - 0.01x^2 \)
   approximately \((-1020, -18,353)\)
   and \( (20, 353) \); The solutions indicate the number of magnet sets for which Austin's cost of making the magnets will equal his income from selling them. It does not make sense for Austin to make a negative number of magnet sets, so the solution with a negative \( x \)-value can be ignored. The solution \((20, 353)\) shows that if Austin makes and sells approximately 20 magnet sets, his cost of making the sets and his income from selling the sets both are about $353.

### Lesson 13-2
Solve each system algebraically. Check your answers by substituting each solution into one of the original equations. Show your work.

16. \( \begin{align*}
y &= x - 7 \\
y &= -x^2 - 2x - 7
\end{align*} \)
17. \( \begin{align*}
y &= 2x^2 - 12x + 26 \\
y &= 8x - 24
\end{align*} \)
18. \( \begin{align*}
y &= -3(x - 4)^2 + 2 \\
y &= 6x - 31
\end{align*} \)
19. \( \begin{align*}
y &= -0.5x - 1 \\
y &= 0.5x^2 + 3x - 5
\end{align*} \)

A map of a harbor is laid out on a coordinate grid, with the origin marking a buoy at the center of the harbor. A fishing boat is following a path that can be represented on the map by the equation \( y = x^2 - 2x - 4 \). A ferry is following a linear path that passes through the points \((-3, 7)\) and \((0, -5)\) when represented on the map. Use this information for Items 20–22.

20. Write a system of equations that can be used to determine whether the paths of the boats will cross.
21. Use substitution to solve the system.
22. Interpret the solution(s) of the system in the context of the situation.
23. How many real solutions does the following system have?

\( \begin{align*}
y &= -x^2 + 4x \\
y &= 3x + 5
\end{align*} \)
   A. none  B. one  C. two  D. infinitely many
24. Explain how you can support your answer to Item 23 algebraically.

### ADDITIONAL PRACTICE
If students need more practice on the concepts in this activity, see the Teacher Resources at SpringBoard Digital for additional practice problems.
During a Boston Red Sox baseball game at Fenway Park, the opposing team hit a home run over the left field wall. An unhappy Red Sox fan caught the ball and threw it back onto the field. The height of the ball, $h(t)$, in feet, $t$ seconds after the fan threw the baseball, is given by the function $h(t) = -16t^2 + 32t + 48$.

1. Graph the equation on the coordinate grid below.

![Green Monster Graph](image)

2. Find each measurement value described below. Then tell how each value relates to the graph.
   a. At what height was the fan when he threw the ball?
   b. What was the maximum height of the ball after the fan threw it?
   c. When did the ball hit the field?

3. What are the reasonable domain and reasonable range of $h(t)$? Explain how you determined your answers.

4. Does the baseball reach a height of 65 ft? Explain your answer both graphically and algebraically.

5. Each baseball team in a minor league plays each other team three times during the regular season.
   a. The table shows the relationship between the number of teams in a baseball league and the total number of games required for each team to play each of the other teams three times. Write a quadratic equation that models the data in the table.
   b. Last season, the total number of games played in the regular season was 35 more than 10 times the number of teams. Use this information to write a linear equation that gives the number of regular games $y$ in terms of the number of teams $x$.
   c. Write a system of equations using the quadratic equation from part a and the linear equation from part b. Then solve the system and interpret the solutions.

### Common Core State Standards for Embedded Assessment 3

**HSA-REI.D.11** Explain why the $x$-coordinates of the points where the graphs of the equations $y = f(x)$ and $y = g(x)$ intersect are the solutions of the equation $f(x) = g(x)$; find the solutions approximately, e.g., using technology to graph the functions, make tables of values, or find successive approximations. Include cases where $f(x)$ and/or $g(x)$ are linear, polynomial, rational, or absolute value.

**HSF-IF.B.4** For a function that models a relationship between two quantities, interpret key features of graphs and tables in terms of the quantities, and sketch graphs showing key features given a verbal description of the relationship.

**HSF-IF.B.5** Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes.

---

**Answer Key**

2. a. 48 ft; The $y$-intercept represents the height of the ball when it was thrown.
   b. 64 ft; The $y$-coordinate of the vertex of the graph represents the maximum height of the ball.
   c. 3 s after the ball was thrown; The positive $x$-intercept indicates when the height of the ball is 0 ft. In other words, it shows the time at which the ball hits the ground.

3. Negative values for the time do not make sense, and the ball hits the ground after 3 s, so a reasonable domain for the function is $0 \leq t \leq 3$. Negative values for the height do not make sense, and the maximum height the ball reaches is 64 ft, so a reasonable range is $0 \leq h \leq 64$.

4. No. Graphically: The vertex of the graph is (1, 64), so the maximum height of the ball is 64 ft.
   Algebraically: Set $h(t)$ equal to 65: $65 = -16t^2 + 32t + 48$. Subtract 65 from both sides to write the equation in standard form: $0 = -16t^2 + 32t - 17$. The Quadratic Formula shows that $t = 1 \pm \frac{i}{4}$. Because the equation has complex solutions, there is no real value of $t$ that results in a height of 65 ft.

5. a. $y = 1.5x^2 - 1.5x$
   b. $y = 10x + 35$
   c. $y = 1.5x^2 - 1.5x; (\frac{7}{3}, \frac{35}{3})$ and $(10, 135);$ The $x$-coordinates of the solutions represent the number of teams in the league. Only a positive, whole number of teams makes sense, so the solution $(\frac{7}{3}, \frac{35}{3})$ can be ignored. The solution $(10, 135)$ indicates that the league has 10 teams and the teams played a total of 135 games.
### Embedded Assessment 3
#### Use after Activity 13

#### Embedded Assessment 3

**Graphing Quadratic Functions and Solving Systems**

**THE GREEN MONSTER**

#### Scoring Guide

<table>
<thead>
<tr>
<th>Mathematics Knowledge and Thinking (Items 2, 4, 5)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Effective understanding of how to solve quadratic equations and systems of equations</td>
<td>• Adequate understanding of how to solve quadratic equations and systems of equations</td>
<td>• Partial understanding of how to solve quadratic equations and systems of equations</td>
<td>• Inaccurate or incomplete understanding of how to solve quadratic equations and systems of equations</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of how to write linear and quadratic models from verbal descriptions or tables of data</td>
<td>• Largely correct understanding of how to write linear and quadratic models from verbal descriptions or tables of data</td>
<td>• Partial understanding of how to write linear and quadratic models from verbal descriptions or tables of data</td>
<td>• Little or no understanding of how to write linear and quadratic models from verbal descriptions or tables of data</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of how to use an equation or graph to identify key features of a quadratic function</td>
<td>• Largely correct understanding of how to use an equation or graph to identify key features of a quadratic function</td>
<td>• Difficulty with using an equation or graph to identify key features of a quadratic function</td>
<td>• Little or no understanding of how to use an equation or graph to identify key features of a quadratic function</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem Solving (Items 2, 4, 5c)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• An appropriate and efficient strategy that results in a correct answer</td>
<td>• A strategy that may include unnecessary steps but results in a correct answer</td>
<td>• A strategy that results in some incorrect answers</td>
<td>• No clear strategy when solving problems</td>
<td></td>
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</tbody>
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<table>
<thead>
<tr>
<th>Mathematical Modeling / Representations (Items 1, 2, 3, 4, 5)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Effective understanding of how to interpret solutions to a system of equations that represents a real-world scenario</td>
<td>• Adequate understanding of how to interpret solutions to a system of equations that represents a real-world scenario</td>
<td>• Partial understanding of how to interpret solutions to a system of equations that represents a real-world scenario</td>
<td>• Little or no understanding of how to interpret solutions to a system of equations that represents a real-world scenario</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range</td>
<td>• Largely correct understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range</td>
<td>• Some difficulty with modeling real-world scenarios with quadratic and linear functions, including reasonable domain and range</td>
<td>• Inaccurate or incomplete understanding of how to model real-world scenarios with quadratic and linear functions, including reasonable domain and range</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario</td>
<td>• Largely correct understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario</td>
<td>• Some difficulty with graphing and interpreting key features of a quadratic function that represents a real-world scenario</td>
<td>• Inaccurate or incomplete understanding of how to graph and interpret key features of a quadratic function that represents a real-world scenario</td>
<td></td>
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</tbody>
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<thead>
<tr>
<th>Reasoning and Communication (Items 2, 3, 4)</th>
<th>Exemplary</th>
<th>Proficient</th>
<th>Emerging</th>
<th>Incomplete</th>
</tr>
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<tbody>
<tr>
<td>• Precise use of appropriate math terms and language to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario</td>
<td>• Adequate explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario</td>
<td>• Misleading or confusing explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario</td>
<td>• Incomplete or inaccurate explanations to relate the features of a quadratic model, including reasonable domain and range, to a real-world scenario</td>
<td></td>
</tr>
<tr>
<td>• Clear and accurate use of mathematical work to explain whether or not the height could reach 65 feet</td>
<td>• Correct use of mathematical work to explain whether or not the height could reach 65 feet</td>
<td>• Partially correct explanation of whether or not the height could reach 65 feet</td>
<td>• Incorrect or incomplete explanation of whether or not the height could reach 65 feet</td>
<td></td>
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</tbody>
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